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FOURIER TRANSFORMS ON A SEMISIMPLE LIE ALGEBRA I.*

By HARISH-CHANDRA.

1. Introduction. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$. Consider the connected adjoint group G of $\mathfrak{g}_0$ and its Cartan subgroup A corresponding to $\mathfrak{h}_0$ (see [2(e), §2]). Put $G^* = G/A$ and $x^*H = xH$ $(x \in G, H \in \mathfrak{h}_0)$, where $x \to x^*$ is the natural mapping of G on G^*. Let dx^* denote the invariant measure on G^*. Define π as in [2(h), §3] and let $\mathfrak{h}_0'$ be the set of those points $H \in \mathfrak{h}_0$ where $\pi(H) \neq 0$. Then we shall show that the integral

$$\phi_f(H) = \pi(H) \int_{G^*} f(x^*H)\, dx^* \qquad (H \in \mathfrak{h}_0')$$

is convergent for $f \in \mathcal{C}(\mathfrak{g}_0)$ (see [2(h), §2] for the notation used here and below.). Moreover, if $\tilde{f}$ denotes the Fourier transform of f, there exists a rather simple relationship between $\phi_{\tilde{f}}$ and ϕ_f (see Lemma 24). This fact is the appropriate generalization to the noncompact case of Theorem 3 of [2(h)]. Let $\mathfrak{h}_1$ be a connected component of $\mathfrak{h}_0'$. One of our principal objects is to show that

$$\lim_{H \to 0} \phi_f(H; \partial(\pi)) = cf(0) \qquad (H \in \mathfrak{h}_1, f \in \mathcal{C}(\mathfrak{g}_0)),$$

where c is a complex number independent of f. (This corresponds to Lemma 16 of [2(h)] in the compact case). Actually, it would turn out that $c = 0$ very often. Nevertheless, $\mathfrak{h}_0$ and $\mathfrak{h}_1$ can always be so chosen that $c \neq 0$. (The proof of these statements is rather long and will be completed only in the next paper of this series.) As we shall see in another paper, these results are intimately related to the Plancherel formula for any connected Lie group which is locally isomorphic to G. In the special case of the $n \times n$ real unimodular group, the above expression for $f(0)$ in terms of ϕ_f has been obtained by Gelfand and Graev [1].

Let $\mathfrak{k}_0$ be the subalgebra of $\mathfrak{g}_0$ corresponding to a maximal compact subgroup K of G. We shall see that for most of our proofs, the crucial case is when $\mathfrak{h}_0 \subset \mathfrak{k}_0$. Lemma 5 contains the fundamental inequalities which

* Received July 30, 1956.

allow us to establish the main properties of ϕ_f (see Theorem 2) in this case. In Section 5, we discard the above assumption on $\mathfrak{h}_0$ and extend these results to the general case. The most important one among them is the relation $\phi_{\partial(p)f} = \partial(\tilde{p})\phi_f$ $(p \in I(\mathfrak{g}), f \in \mathcal{B}(\mathfrak{g}_0))$ which, in its turn, follows from Theorem 1 of [2(h)]. On the other hand, these differential equations give us the above-mentioned results concerning Fourier transforms (see Section 6).

Select $\mathfrak{h}_0$ and $\mathfrak{h}_1$ as above, and put

$$T'(f) = \lim_{H \to 0} \phi_{\tilde{f}}(H; \partial(\pi)) \qquad (H \in \mathfrak{h}_1, f \in \mathcal{B}(\mathfrak{g}_0)).$$

Then T' is a $\mathcal{B}$-distribution on $\mathfrak{g}_0$ and we have to show that $T' = c$, where c is a constant. In Section 7, we shall obtain the weaker result that T' coincides with a constant on each connected component of the set of regular elements in $\mathfrak{g}_0$. The problem is thus reduced (see the corollary to Lemma 29) to proving that the various constants so obtained are all equal. This requires a deeper investigation which will be carried out in another paper.

In Section 8, we define the notion of a fundamental Cartan subalgebra and prove that any two such algebras are conjugate under G. Moreover, we show that

$$\lim_{H \to 0} \phi_f(H; \partial(\pi)) = 0 \qquad (H \in \mathfrak{h}_1, f \in \mathcal{B}(\mathfrak{g}_0))$$

if $\mathfrak{h}_0$ is not fundamental. Thus it only remains to study the case when $\mathfrak{h}_0$ is fundamental. In particular, this is so if $\mathfrak{h}_0 \subset \mathfrak{k}_0$, and so we again make this assumption. The principal result of Section 9 is contained in Theorem 4, and its proof is based on Theorem 5 of [2(h)]. The rest of this paper is devoted to proving that the constant c_0 of Theorem 4 is not zero. For this, we have to make use of some (unpublished) work of de Rham on the elementary solutions of a certain type of differential operators. (Similar results also appear to have been obtained by Gelfand and Graev [1].) We apply de Rham's formula to the operator $\partial(\omega)$, where ω is the Casimir polynomial of $\mathfrak{g}_0$. This enables us to prove (see Lemma 41) the existence of a connected component $\mathfrak{h}_1$ of $\mathfrak{h}_0'$ and of a function $g \in \mathcal{B}(\mathfrak{g}_0)$ such that

$$\lim_{H \to 0} \phi_g(H; \partial(\pi)) \neq 0 \qquad (H \in \mathfrak{h}_1).$$

We shall need this fact in our study of the general case when $\mathfrak{h}_0$ is fundamental but the corresponding Cartan subgroup is not necessarily compact.

The main results of Theorem 3 have been announced in a short note [2(i)] (see also [2(j)]).

2. A preliminary result. Let R and C be the fields of real and complex numbers, respectively, E_0 a vector space over R of finite dimension and F_0 a subspace of E_0. Let E be the complexification of E_0 and F the complexification of F_0 in E. We identify (real-valued) linear functions on E_0 with their linear extensions on E. Let E' be the space of all such functions. Then elements of E' will be called *real* linear functions on E. Let F' be the corresponding space for F. Then E' and F' are vector spaces over R. For any $\lambda \in E'$, let $\bar{\lambda}$ denote the restriction of λ on F. Then $\lambda \to \bar{\lambda}$ is a linear mapping of E' into F' whose kernel we denote by E_1'. Suppose we have introduced orders [2(f), §2] in E' and F'. Then these orders are said to be compatible if $\bar{\lambda} > 0$ implies $\lambda > 0$ ($\lambda \in E'$). Given any order in F', we can always define an order on E' compatible with it as follows. Select any order on E_1' and call an element $\lambda \in E'$ positive if either $\bar{\lambda}$ is positive in F', or $\lambda \in E_1'$ and it is positive in the order chosen in E_1'.

Let $\mathfrak{g}_0$ be a semisimple Lie algebra over R. Define $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as in [2(c), §2] and let $\mathfrak{a}_{\mathfrak{p}_0}$ be a maximal abelian subspace of $\mathfrak{p}_0$. We extend it to a Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}_0$. Then $\mathfrak{a}_0 = \mathfrak{a}_{\mathfrak{p}_0} + \mathfrak{a}_{\mathfrak{k}_0}$ where $\mathfrak{a}_{\mathfrak{k}_0} = \mathfrak{a}_0 \cap \mathfrak{k}_0$ (see [2(c), p. 188]). Complexify $\mathfrak{g}_0$ to $\mathfrak{g}$ and let $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}_{\mathfrak{p}}$, $\mathfrak{a}$, $\mathfrak{a}_{\mathfrak{k}}$ denote the complexifications of $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_{\mathfrak{p}_0}$, $\mathfrak{a}_0$, $\mathfrak{a}_{\mathfrak{k}_0}$, respectively, in $\mathfrak{g}_0$. Now take[1] $E_0 = \mathfrak{a}_{\mathfrak{p}_0} + (-1)^{\frac{1}{2}}\mathfrak{a}_{\mathfrak{k}_0}$ and $F_0 = \mathfrak{a}_{\mathfrak{p}_0}$ in the above set up, and select compatible orders in the spaces of real linear functions on $\mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{p}}$, respectively. Let $(\alpha_1, \cdots, \alpha_l)$ be a fundamental system [2(f), §2] of positive roots (of $\mathfrak{g}$ with respect to $\mathfrak{a}$) in our order. Then if $p = \dim \mathfrak{a}_{\mathfrak{p}} = \dim_R \mathfrak{a}_{\mathfrak{p}_0}$, we have the following result.

LEMMA 1. *It is possible to select p roots, say $(\alpha_1, \cdots, \alpha_p)$, among $(\alpha_1, \cdots, \alpha_l)$ with the following two properties.*[2]

(1) *$\bar{\alpha}_1, \cdots, \bar{\alpha}_p$ are linearly independent.*

(2) *If α is any root, $\bar{\alpha} = m_1\bar{\alpha}_1 + \cdots + m_p\bar{\alpha}_p$ where m_i are integers which are either all nonnegative or all nonpositive.*

Let Q be the set of all positive roots of $\mathfrak{g}$ in our order and Q_+ the subset consisting of those $\alpha \in Q$ for which $\bar{\alpha} \neq 0$. Select a minimal subset $(\alpha_1, \cdots, \alpha_q)$ (say) of our fundamental system such that, for every $\alpha \in Q_+$, $\bar{\alpha}$ can be written in the form $\bar{\alpha} = m_1\bar{\alpha}_1 + \cdots + m_q\bar{\alpha}_q$, where m_i are all nonnegative integers. Then it is enough to prove that $\bar{\alpha}_1, \cdots, \bar{\alpha}_q$ are linearly independent. This requires a few lemmas.

---

[1] We select once for all a square root of -1 in C and denote by $(-1)^{k/2}$ its k-th power for any integer k.

[2] The bar denotes restriction on $\mathfrak{a}_{\mathfrak{p}}$.

LEMMA 2. *If i, j are two distinct indices $(1 \leqq i, j \leqq q)$, then $\tilde{\alpha}_i - \tilde{\alpha}_j \neq 0$ or $\tilde{\alpha}$ for any $\alpha \in Q_+$.*

Without loss of generality, we may assume that $i = 1$, $j = 2$. In view of the minimality of our system $(\alpha_1, \cdots, \alpha_q)$, it is obvious that $\tilde{\alpha}_1 - \tilde{\alpha}_2 \neq 0$. So let us now suppose that $\tilde{\alpha}_1 = \tilde{\alpha} + \tilde{\alpha}_2$ for some $\alpha \in Q_+$. Then $\tilde{\alpha}_1 = (m_1\tilde{\alpha}_1 + \cdots + m_q\tilde{\alpha}_q) + \tilde{\alpha}_2$ where $m_1, \cdots, m_q$ are nonnegative integers. Hence

$$(m_1 - 1)\tilde{\alpha}_1 + (m_2 + 1)\tilde{\alpha}_2 + \sum_{3 \leqq k \leqq q} m_k\tilde{\alpha}_k = 0.$$

It is clear that $m_1 = 0$, for otherwise, since $\tilde{\alpha}_k > 0$ $(1 \leqq k \leqq q)$, the left side would be positive. Therefore

$$\tilde{\alpha}_1 = (m_2 + 1)\tilde{\alpha}_2 + \sum_{3 \leqq k \leqq q} m_k\tilde{\alpha}_k,$$

and we get a contradiction with the minimality of the system $(\alpha_1, \cdots, \alpha_q)$.

Let B denote the fundamental bilinear form on $\mathfrak{g}$ [2(c), §2] and $\tilde{\theta}$ the conjugation with respect to the compact real form $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$. Then $\| X \|^2 = -B(X, \tilde{\theta}(X))$ $(X \in \mathfrak{g})$ is a positive-definite Hermitian form on $\mathfrak{g}$. Since the restriction of B on $\mathfrak{a}_{\mathfrak{p}_0}$ is real and nondegenerate, for each real linear function λ on $\mathfrak{a}_{\mathfrak{p}}$, there exists a unique element $\Gamma_\lambda \in \mathfrak{a}_{\mathfrak{p}_0}$ such that $B(\Gamma, \Gamma_\lambda) = \lambda(\Gamma)$ for all $\Gamma \in \mathfrak{a}_{\mathfrak{p}}$. Define θ as in [2(c), §2].

LEMMA 3. *Suppose λ is a real linear function on $\mathfrak{a}_{\mathfrak{p}}$ and X an element in $\mathfrak{g}_0$ such that $[\Gamma, X] = \lambda(\Gamma)X$ for all $\Gamma \in \mathfrak{a}_{\mathfrak{p}}$. Then $[\theta(X), X] = \| X \|^2 \Gamma_\lambda$.*

Let $X = Y + Z$ $(Y \in \mathfrak{p}_0, Z \in \mathfrak{k}_0)$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, it follows that $[\Gamma, Y] = \lambda(\Gamma)Z$, $[\Gamma, Z] = \lambda(\Gamma)Y$ $(\Gamma \in \mathfrak{a}_{\mathfrak{p}})$. Moreover, $[\theta(X), X] = 2[Z, Y]$, and so it is clear that $[\theta(X), X]$ lies in $\mathfrak{p}_0$ and it commutes with every element in $\mathfrak{a}_{\mathfrak{p}_0}$. Therefore, since $\mathfrak{a}_{\mathfrak{p}_0}$ is maximal abelian in $\mathfrak{p}_0$, $[\theta(X), X] \in \mathfrak{a}_{\mathfrak{p}_0}$. Finally, if $\Gamma \in \mathfrak{a}_{\mathfrak{p}}$,

$$B(\Gamma, [\theta(X), X]) = -B([\Gamma, X], \theta(X)) = -\lambda(\Gamma)B(X, \theta(X)) = \lambda(\Gamma)\| X \|^2,$$

and this prove that $[\theta(X), X] = \| X \|^2 \Gamma_\lambda$.

For any two linear functions λ and μ on $\mathfrak{a}_{\mathfrak{p}}$, put $\langle \lambda, \mu \rangle = \lambda(\Gamma_\mu) = B(\Gamma_\lambda, \Gamma_\mu)$ and $|\lambda| = \langle \lambda, \lambda \rangle^{\frac{1}{2}} \geqq 0$. Then if $\lambda \neq 0$, $|\lambda| = \| \Gamma_\lambda \| > 0$. Let Σ be the set of all linear functions λ on $\mathfrak{a}_{\mathfrak{p}}$ for which there exists an element $X_\lambda \neq 0$ in $\mathfrak{g}_0$ with the property that $[\Gamma, X_\lambda] = \lambda(\Gamma)X_\lambda$ for all $\Gamma \in \mathfrak{a}_{\mathfrak{p}}$. It is obvious that Σ consists of zero and $\pm\tilde{\alpha}$ $(\alpha \in Q_+)$.

LEMMA 4. *Suppose $\lambda, \mu \in \Sigma$ and $\lambda - \mu \notin \Sigma$. Then $\langle \lambda, \mu \rangle \leqq 0$.*

Select nonzero elements X_λ, X_μ in $\mathfrak{g}_0$ as above. We may assume that $\| X_\mu \| = 1$. It follows from our assumption that neither λ nor μ is zero. Put $X = 2^{\frac{1}{2}} |\mu|^{-1} X_\mu$, $Y = \theta(X)$ and $\Gamma_0 = 2|\mu|^{-2}\Gamma_\mu$. Then $[Y, X] = \Gamma_0$, $[\Gamma_0, X] = 2X$, $[\Gamma_0, Y] = -2Y$. Hence $\mathfrak{l}_0 = RX + RY + R\Gamma_0$ is a three-dimensional Lie algebra and $Z \to \operatorname{ad} Z$ $(Z \in \mathfrak{l}_0)$ is a representation of $\mathfrak{l}_0$ on $\mathfrak{g}_0$. Now $(\operatorname{ad}\Gamma_0)X_\lambda = \lambda(\Gamma_0)X_\lambda$ and $(\operatorname{ad} Y)X_\lambda = 0$ since $\lambda - \mu \notin \Sigma$. Therefore it follows immediately from the theory of finite-dimensional representations of $\mathfrak{l}_0$ (see [2(a), Lemma 1]) that $\lambda(\Gamma_0) \leqq 0$. But $\lambda(\Gamma_0) = 2|\mu|^{-2}\lambda(\Gamma_\mu)$, and so we conclude that $\langle \lambda, \mu \rangle \leqq 0$.

COROLLARY. *If* $i \neq j$ $(1 \leqq i, j \leqq q)$ *then* $\langle \tilde{\alpha}_i, \tilde{\alpha}_j \rangle \leqq 0$.

This is an immediate consequence of Lemmas 2 and 4.

Now we can prove that $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_q$ are linearly independent. For otherwise, since they are real, we could select $a_i \in R$ $(1 \leqq i \leqq q)$ not all zero such that $a_1\tilde{\alpha}_1 + \cdots + a_q\tilde{\alpha}_q = 0$. We may suppose that $a_1, \cdots, a_k > 0$ while $a_{k+1}, \cdots, a_q \leqq 0$. Then if $b_j = -a_j$ $(k < j \leqq q)$,

$$a_1\tilde{\alpha}_1 + \cdots + a_k\tilde{\alpha}_k = b_{k+1}\tilde{\alpha}_{k+1} + \cdots + b_q\tilde{\alpha}_q = \lambda \quad \text{(say)}.$$

Therefore

$$|\lambda|^2 = \sum_{1 \leqq i \leqq k} \sum_{k < j \leqq q} a_i b_j \langle \tilde{\alpha}_i, \tilde{\alpha}_j \rangle \leqq 0$$

from the above corollary. This implies that $\lambda = a_1\tilde{\alpha}_1 + \cdots + a_k\tilde{\alpha}_k = 0$ which, however, is impossible since $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_k$ are all positive and $a_1, \cdots, a_k > 0$. This completes the proof of Lemma 1.

3. Some inequalities. The positive-definite Hermitian form

$$\| X \|^2 = -B(X, \theta(X)) \qquad (X \in \mathfrak{g})$$

defines the structure of a Hilbert space on $\mathfrak{g}$. For any linear transformation T in $\mathfrak{g}$, put $\| T \|^2 = \operatorname{sp}(T^*T)$, where T^* is the adjoint of T with respect to this structure. It is clear that $(\operatorname{ad} X)^* = -\operatorname{ad}(\theta(X))$ $(X \in \mathfrak{g})$, and therefore $\| \operatorname{ad} X \| = \| X \|$. Let G be the connected adjoint group of $\mathfrak{g}_0$ and K the analytic subgroup of G corresponding to $\mathfrak{k}_0$. Then every element $k \in K$ is a unitary transformation on $\mathfrak{g}$, and therefore $\| k \| = 1$.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. *Until the end of Section* 4 *we shall assume that* $\mathfrak{h}_0 \subset \mathfrak{k}_0$. Let $\mathfrak{h}$ denote the complexification of $\mathfrak{h}_0$ in $\mathfrak{g}$ and P the set of all positive roots (under some arbitrary but fixed order) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let P_- and P_0 be the subsets of P consisting of the compact and the noncompact roots, respectively (see [2(f), p. 751]). Put $\pi_0 = \prod_{\alpha \in P_0} \alpha$. Then π_0 is a polynomial function on $\mathfrak{h}$.

LEMMA 5. *Suppose $x \in G$ and $H \in \mathfrak{h}_0$. Then*

$$\| xH + \theta(xH) \|^2 = 2 \| H \|^2 + 2 \| xH \|^2.$$

Moreover, there exist integers $m, q \geqq 0$ and a positive number c such that

$$\| x \| \, | \pi_0(H) |^q \leqq c \, \| xH + \theta(xH) \|^m$$

for all $x \in G$ and $H \in \mathfrak{h}_0$.

K being compact [5], any maximal abelian subalgebra of $\mathfrak{k}_0$ is conjugate to $\mathfrak{h}_0$ under K. Hence we can select $k \in K$ such that $k(\mathfrak{a}_{\mathfrak{k}_0}) \subset \mathfrak{h}_0$. Then $k(\mathfrak{a}_{\mathfrak{p}_0})$ is also a maximal abelian subspace of $\mathfrak{p}_0$, and therefore, by replacing $\mathfrak{a}_0$ by $k\mathfrak{a}_0$, we may assume that $\mathfrak{a}_{\mathfrak{k}_0} \subset \mathfrak{h}_0$. Let $A_{\mathfrak{p}}$ be the analytic subgroup of G corresponding to $\mathfrak{a}_{\mathfrak{p}_0}$. Then $G = KA_{\mathfrak{p}}K$ (see [2(g), §12]), and every element in K is a unitary transformation on $\mathfrak{g}$ which commutes with θ. Therefore for the proof of our lemma, it is obviously enough to consider the case when $x = ak$ $(a \in A_{\mathfrak{p}}, k \in K)$. Fix an element $H \in \mathfrak{h}_0$, and put $kH = X$. Then $xH = aX$ and $\theta(xH) = a^{-1}X$. Hence $\| xH + \theta(xH) \| = \| aX + a^{-1}X \|$. But since a is self-adjoint, it is obvious that $\| aX + a^{-1}X \|^2 = \| aX \|^2 + \| a^{-1}X \|^2 + 2 \| X \|^2$. Moreover, $\| a^{-1}X \| = \| \theta(aX) \| = \| aX \|$, and therefore

$$\| xH + \theta(xH) \|^2 = 2 \| X \|^2 + 2 \| aX \|^2 = 2 \| H \|^2 + 2 \| xH \|^2.$$

This proves the first assertion. The proof of the second statement is, however, more complicated.

Define Q and Q_+ as in Section 2, and for any root β of $\mathfrak{g}$ with respect to $\mathfrak{a}$, choose an element $X_\beta \neq 0$ in $\mathfrak{g}$ such that $[\Gamma, X_\beta] = \beta(\Gamma) X_\beta$ for all $\Gamma \in \mathfrak{a}$. Then the function $\theta\beta : \Gamma \to \beta(\theta\Gamma)$ is also a root.[3] Let Q_- be the complement of Q_+ in Q. Then if $\beta \in Q_-$, $\theta\beta = \beta$ and $X_\beta, X_{-\beta} \in \mathfrak{k}$. On the other hand $\theta\beta < 0$ if $\beta \in Q_+$. Put $\mathfrak{q} = \sum_{\beta \in Q_+} CX_\beta$, $\mathfrak{q}^- = \sum_{\beta \in Q_+} CX_{-\beta}$, and let $\mathfrak{l}$ be the centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$. Then $\mathfrak{l} \supset \mathfrak{a}_{\mathfrak{k}}$ and

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{l} + \mathfrak{a}_{\mathfrak{p}} + \mathfrak{q} = \mathfrak{k} + \mathfrak{a}_{\mathfrak{p}} + \mathfrak{q} = \mathfrak{p} + \mathfrak{l} + \mathfrak{q},$$

where all the sums are direct.[3] Therefore

$$\dim \mathfrak{k} = \dim \mathfrak{g} - \dim \mathfrak{a}_{\mathfrak{p}} - \dim \mathfrak{q} = \dim \mathfrak{l} + \dim \mathfrak{q}.$$

Define a linear mapping ϕ of $\mathfrak{l} + \mathfrak{q}$ into $\mathfrak{k}$ by $\phi(X) = X + \theta(X)$ $(X \in \mathfrak{l} + \mathfrak{q})$. Since $(\mathfrak{l} + \mathfrak{q}) \cap \mathfrak{p} = \{0\}$, the kernel of ϕ is zero. Therefore since $\mathfrak{k}$ and $\mathfrak{l} + \mathfrak{q}$ have the same dimension, ϕ is an isomorphism of $\mathfrak{l} + \mathfrak{q}$ onto $\mathfrak{k}$.

[3] All the facts stated here are well known (see [2(c), p. 188]).

Now if $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}$, $\operatorname{ad}\Gamma$ is a self-adjoint operator on $\mathfrak{g}$, and its eigenvalues are zero and $\pm\beta(\Gamma)$ $(\beta \in Q_+)$. Therefore

$$\| (\operatorname{ad}\Gamma)^2 Z \| \geqq \lambda(\Gamma) \| (\operatorname{ad}\Gamma) Z \| \qquad (Z \in \mathfrak{g}, \Gamma \in \mathfrak{a}_{\mathfrak{p}_0}),$$

where $\lambda(\Gamma) = \min |\beta(\Gamma)|$ and β runs over all roots in Q_+ for which $\beta(\Gamma) \neq 0$. (In case $\Gamma = 0$, we put $\lambda(\Gamma) = 0$.)

Now select[4] $\alpha_1, \cdots, \alpha_p \in Q_+$ according to Lemma 1, and choose $\Gamma_i \in \mathfrak{a}_{\mathfrak{p}_0}$ $(1 \leqq i \leqq p)$ such that[5] $\alpha_i(\Gamma_j) = \delta_{ij}$ $(1 \leqq i, j \leqq p)$. Then $(\Gamma_1, \cdots, \Gamma_p)$ is a base for $\mathfrak{a}_{\mathfrak{p}_0}$ over R, and it is clear that $\lambda(\Gamma_i) \geqq |\alpha_i(\Gamma_i)| = 1$. Therefore

$$\| (\operatorname{ad}\Gamma_i)^2 Z \| \geqq \| (\operatorname{ad}\Gamma_i) Z \| \qquad (Z \in \mathfrak{g}, 1 \leqq i \leqq p).$$

Let Y be an element in $\mathfrak{q}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. Then $\operatorname{ad} Y$ is nilpotent and $B(Y, Y) = B(\theta(Y), \theta(Y)) = 0$. Therefore

$$\| \phi(Y) \|^2 = -2B(Y, \theta(Y)) = 2 \| Y \|^2.$$

Hence if $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$ and $X \in \mathfrak{l}_0 + \mathfrak{q}_0$,

$$\| \phi((\operatorname{ad}\Gamma_i)^2 X) \|^2 = 2 \| (\operatorname{ad}\Gamma_i)^2 X \|^2$$

since $(\operatorname{ad}\Gamma_i)^2 X \in \mathfrak{q}_0$. Moreover, $(\operatorname{ad}\Gamma_i)^2$ commutes with θ. Therefore

$$\| \phi((\operatorname{ad}\Gamma_i)^2 X) \| = \| (\operatorname{ad}\Gamma_i)^2 \phi(X) \| \geqq \| (\operatorname{ad}\Gamma_i) \phi(X) \|$$

as we have seen above. Hence

$$\| (\operatorname{ad}\Gamma_i)^2 X \| \geqq 2^{-\frac{1}{2}} \| (\operatorname{ad}\Gamma_i) \phi(X) \| \qquad (X \in \mathfrak{l}_0 + \mathfrak{q}_0, 1 \leqq i \leqq p).$$

On the other hand, we know (see Iwasawa [3]) that $\mathfrak{g}_0$ is the direct sum of $\mathfrak{k}_0$, $\mathfrak{a}_{\mathfrak{p}_0}$ and $\mathfrak{q}_0$, so that

$$\begin{aligned} \dim_R \mathfrak{q}_0 &= \dim_R \mathfrak{g}_0 - \dim_R \mathfrak{k}_0 - \dim_R \mathfrak{a}_{\mathfrak{p}_0} \\ &= \dim \mathfrak{g} - \dim \mathfrak{k} - \dim \mathfrak{a}_{\mathfrak{p}} = \dim \mathfrak{q}. \end{aligned}$$

Therefore $\dim_R (\mathfrak{l}_0 + \mathfrak{q}_0) = \dim \mathfrak{l} + \dim \mathfrak{q} = \dim \mathfrak{k} = \dim_R \mathfrak{k}_0$. Since $\phi(\mathfrak{l}_0 + \mathfrak{q}_0)$ is obviously contained in $\mathfrak{g}_0$, it follows that ϕ defines a linear isomorphism of $\mathfrak{l}_0 + \mathfrak{q}_0$ onto $\mathfrak{k}_0$. Let ψ denote the inverse of this isomorphism. Then the above result may be written in the form

$$\| (\operatorname{ad}\Gamma_i)^2 \psi(X) \| \geqq 2^{-\frac{1}{2}} \| (\operatorname{ad} X) \Gamma_i \| \qquad (X \in \mathfrak{k}_0, 1 \leqq i \leqq p).$$

Let $(\operatorname{ad} X)_{\mathfrak{p}}$ denote the restriction of $\operatorname{ad} X$ on $\mathfrak{p}$ $(X \in \mathfrak{k}_0)$, and put $\mu(X) = |\gamma|$,

[4] All our lemmas are true trivially when $\mathfrak{g}_0 = \mathfrak{k}_0$. So whenever it is convenient for the proof, we shall assume that $\mathfrak{p}_0 \neq \{0\}$.

[5] $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$.

where γ is an eigenvalue of $(\mathrm{ad}\, X)_{\mathfrak{p}}$ with the smallest possible absolute value. Then since $\mathrm{ad}\, X$ is skew-adjoint, it is obvious that $\| (\mathrm{ad}\, X) Y \| \geqq \mu(X) \| Y \|$ $(Y \in \mathfrak{p})$. Therefore

$$\prod_{i=1}^{p} \| (\mathrm{ad}\, \Gamma_i)^2 \psi(X) \| \geqq 2^{-p/2} \prod_{1 \leqq i \leqq p} \| (\mathrm{ad}\, X) \Gamma_i \| \geqq 2^{-p/2} \mu(X)^p \prod_{1 \leqq i \leqq p} \| \Gamma_i \|.$$

Moreover it is obvious that $\mu(kX) = \mu(X)$ $(k \in K, X \in \mathfrak{k}_0)$. Hence

$$\prod_{1 \leqq i \leqq p} \| (\mathrm{ad}\, \Gamma_i)^2 \psi(kH) \| \geqq c_0 \mu(H)^p \qquad (H \in \mathfrak{h}_0, k \in K)$$

where $c_0 = 2^{-p/2} \prod_{1 \leqq i \leqq p} \| \Gamma_i \|$ is a positive constant. On the other hand, it follows from the definition of $\mu(H)$ that $\mu(H) = \min_{\beta \in P_0} | \beta(H) |$ $(H \in \mathfrak{h}_0)$. Therefore

$$\prod_{1 \leqq i \leqq p} \| (\mathrm{ad}\, \Gamma_i)^2 \psi(kH) \| \geqq c_0 \min_{\beta \in P_0} | \beta(H) |^p \qquad (H \in \mathfrak{h}_0, k \quad K).$$

Now observe that $| \beta(H) | \leqq \| H \|$ for any $\beta \in P$. Hence if q_0 is the number of roots in P_0,

$$\| H \|^{q_0 - 1} \min_{\beta \in P_0} | \beta(H) | \geqq \prod_{\beta \in P_0} | \beta(H) | = | \pi_0(H) |,$$

and therefore

$$\| H \|^{p(q_0 - 1)} \prod_{1 \leqq i \leqq p} \| (\mathrm{ad}\, \Gamma_i)^2 \psi(kH) \| \geqq c_0 | \pi_0(H) |^p \qquad (H \in \mathfrak{h}_0, k \in K).$$

Let $\mathfrak{a}_{\mathfrak{p}_0}^+$ be the set of all $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}$ such that $\alpha(\Gamma) \geqq 0$ for every $\alpha \in Q$. It is obvious that $\Gamma = t_1 \Gamma_1 + \cdots + t_p \Gamma_p$ $(t_i \in R)$ lies in $\mathfrak{a}_{\mathfrak{p}_0}^+$ if and only if $t_1, \cdots, t_p \geqq 0$. Let $A_{\mathfrak{p}}^+$ be the image of $\mathfrak{a}_{\mathfrak{p}_0}^+$ in $A_{\mathfrak{p}}$ under the exponential mapping. Then it is known (see [2(g), Lemma 38]) that $G = K A_{\mathfrak{p}}^+ K$. Hence in the proof of the second assertion of our lemma, we can assume that $x \in A_{\mathfrak{p}}^+ K$. Select a maximal set $(\alpha_1, \cdots, \alpha_r)$ $(r \geqq p)$ of roots in Q, such that $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_r$ are all distinct. Then it is obvious that $\mathfrak{q} = \sum_{1 \leqq j \leqq r} \mathfrak{q}_j$, where $\mathfrak{q}_j$ is the set of all $Y \in \mathfrak{q}$ for which $[\Gamma, Y] = \alpha_j(\Gamma) Y (\Gamma \in \mathfrak{a}_{\mathfrak{p}})$. Moreover, $\mathrm{ad}\, \Gamma$ is self-adjoint for $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}$, and so it is clear that $\mathfrak{q}_j$ $(1 \leqq j \leqq r)$ are mutually orthogonal. Similarly $\mathfrak{l}$, $\mathfrak{q}$ and $\mathfrak{q}^-$ are orthogonal to each other. Now suppose $x = ak$ $(a \in A_{\mathfrak{p}}^+, k \in K)$ and H is a fixed element in $\mathfrak{h}_0$ such that $\pi_0(H) \neq 0$. Then if $\psi(kH) = L + X$ $(L \in \mathfrak{l}_0, X \in \mathfrak{q}_0)$, we have $kH = 2L + X + \theta(X)$. Hence

$$\begin{aligned} \| xH + \theta(xH) \|^2 &= 2 \| H \|^2 + 2 \| xH \|^2 \\ &= 16 \| L \|^2 + 4 \| X \|^2 + 2 \| aX \|^2 + 2 \| a^{-1} X \|^2. \end{aligned}$$

Put $\nu = \max_{\alpha, i} | \alpha(\Gamma_i) |$ for $\alpha \in Q$ and $1 \leqq i \leqq p$. Then since no eigenvalue of

$\operatorname{ad}\Gamma_i$ can exceed ν in absolute value, $\|(\operatorname{ad}\Gamma_i)^2 Z\| \leqq \nu^2 \|Z\|$ $(Z \in \mathfrak{g}, 1 \leqq i \leqq p)$. Therefore if $a = \exp\Gamma$ $(\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}^+)$ and X_j is the orthogonal projection of X in $\mathfrak{q}_j$, we have

$$\|aX\|^2 \geqq \nu^{-4} \|(\operatorname{ad}\Gamma_i)^2 aX\|^2 = \nu^{-4} \sum_{1 \leqq j \leqq r} e^{2\alpha_j(\Gamma)} \alpha_j(\Gamma_i)^4 \|X_j\|^2 \qquad (1 \leqq i \leqq p).$$

Now we claim that

$$e^{\alpha(\Gamma)} \alpha(\Gamma_i)^2 \geqq e^{\alpha_i(\Gamma)} \alpha(\Gamma_i)^2$$

for $\alpha \in Q_+$. This is obvious if $\alpha(\Gamma_i) = 0$, and so let us suppose that $\alpha(\Gamma_i) \neq 0$. We know that $\tilde{\alpha} = \nu_1 \tilde{\alpha}_1 + \cdots + \nu_p \tilde{\alpha}_p$, where $\nu_j = \alpha(\Gamma_j)$ $(1 \leqq j \leqq p)$ are non-negative integers. Therefore, since $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}^+$ and $\nu_i = \alpha(\Gamma_i) \geqq 1$, we conclude that $\alpha(\Gamma) \geqq \nu_i \alpha_i(\Gamma) \geqq \alpha_i(\Gamma)$, and our inequality follows. This shows that $\|aX\|^2 \geqq \nu^{-4} \sum_{1 \leqq j \leqq r} e^{2\alpha_i(\Gamma)} \alpha_j(\Gamma_i)^4 \|X_j\|^2 = \nu^{-4} e^{2\alpha_i(\Gamma)} \|(\operatorname{ad}\Gamma_i)^2 X\|^2$ $(1 \leqq i \leqq p)$, and therefore

$$\|xH + \theta(xH)\|^p \geqq \|aX\|^p \geqq \nu^{-2p} e^{\gamma(\Gamma)} \prod_{1 \leqq i \leqq p} \|(\operatorname{ad}\Gamma_i)^2 X\|,$$

where $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_p$. On the other hand,

$$\|x\|^2 = \|a\|^2 = \operatorname{sp} a^2 = l + \sum_{\alpha \in Q} (e^{2\alpha(\Gamma)} + e^{-2\alpha(\Gamma)}),$$

where $l = \dim \mathfrak{a}$. Moreover, it is obvious that $\alpha(\Gamma) \leqq \nu\gamma(\Gamma)$ for any $\alpha \in Q$. Therefore if $n = \dim \mathfrak{g}$, $\|x\|^2 \leqq n e^{2\nu\gamma(\Gamma)}$. Also $(\operatorname{ad}\Gamma_i)^2 \psi(kH) = (\operatorname{ad}\Gamma_i)^2 X$. Hence it follows from our earlier result and the first assertion of the lemma that

$$\begin{aligned}
\|x\| &\leqq n^{\frac{1}{2}\nu} 2^{p\nu} \|xH + \theta(xH)\|^{p\nu} \{ \prod_{1 \leqq i \leqq p} \|(\operatorname{ad}\Gamma_i)^2 X\| \}^{-\nu} \\
&\leqq n^{\frac{1}{2}\nu} 2^{p\nu} \|xH + \theta(xH)\|^{p\nu} \{c_0 \,|\, \pi_0(H)\,|^p \|H\|^{-p(q_0-1)}\}^{-\nu} \\
&\leqq n^{\frac{1}{2}\nu} 2^{p\nu} \|xH + \theta(xH)\|^{p\nu} \{c_0 \,|\, \pi_0(H)\,|^p \|xH + \theta(xH)\|^{-p(q_0-1)}\}^{-\nu}.
\end{aligned}$$

So by taking $q = p\nu$, $m = p\nu q_0$ and $c = n^{\frac{1}{2}\nu} 2^{p\nu} c_0^{-\nu}$, we get the second statement of the lemma.

LEMMA 6. *Let λ be any real linear function on $\mathfrak{a}_{\mathfrak{p}}$. Then there exists an integer $m \geqq 0$ such that*

$$e^{\lambda(\Gamma)} \leqq \|\exp\Gamma\|^m$$

for $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}^+$.

It is clear that [4]

$$\|\exp\Gamma\|^2 \geqq \sum_{\alpha \in Q_+} e^{2\alpha(\Gamma)} \geqq r \exp(2r^{-1} \sum_{\alpha \in Q_+} \alpha(\Gamma)) \qquad (\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}^+),$$

where r is the number of roots in Q_+. Moreover, we can obviously choose a positive integer s such that

$$s(\alpha_1(\Gamma) + \alpha_2(\Gamma) + \cdots + \alpha_p(\Gamma)) \geqq \lambda(\Gamma)$$

for $\Gamma \in \mathfrak{a}_{p_0}^+$. Therefore

$$\| \exp \Gamma \|^{rs} \geqq \exp(s \sum_{\alpha \in Q_+} \alpha(\Gamma)) \geqq \exp \lambda(\Gamma) \qquad (\Gamma \in \mathfrak{a}_{p_0}^+),$$

and this proves the lemma.

COROLLARY 1. *Let dx denote the Haar measure of G. Then there exists an integer $s \geqq 0$ such that*

$$\int_G \| x \|^{-s} \, dx < \infty.$$

Let dk and $d\Gamma$, respectively, denote the Haar measure on K and the Euclidean measure on $\mathfrak{a}_{p_0}$. Then it is known (see [2(g), Lemma 38]) that one can normalize dx in such a way that

$$\int_G g(x)\, dx = \int_{K \times \mathfrak{a}_{p_0}^+ \times K} g(k(\exp \Gamma)k')\mu(\Gamma)\, dk\, d\Gamma\, dk' \qquad (g \in C_c(G)).$$

Here $C_c(G)$ is the set of all continuous functions on G which vanish outside a compact set and

$$\mu(\Gamma) = \prod_{\alpha \in Q_+} (e^{\alpha(\Gamma)} - e^{-\alpha(\Gamma)}).$$

It is obvious that

$$\mu(\Gamma) \leqq \exp(\sum_{\alpha \in Q_+} \alpha(\Gamma)) \qquad (\Gamma \in \mathfrak{a}_{p_0}^+).$$

Select a real linar function λ on $\mathfrak{a}_p$ such that $\int_{\mathfrak{a}_{p_0}^+} e^{-\lambda(\Gamma)}\, d\Gamma < \infty$. Then from Lemma 6,

$$\exp(\lambda(\Gamma) + \sum_{\alpha \in Q_+} \alpha(\Gamma)) \leqq \| \exp \Gamma \|^s \qquad (\Gamma \in \mathfrak{a}_{p_0}^+)$$

for some integer s. Hence

$$\mu(\Gamma) \| \exp \Gamma \|^{-s} \leqq e^{-\lambda(\Gamma)},$$

and therefore

$$\int_G \| x \|^{-s}\, dx = \int_{\mathfrak{a}_{p_0}^+} \| \exp \Gamma \|^{-s} \mu(\Gamma)\, d\Gamma \leqq \int_{\mathfrak{a}_{p_0}^+} e^{-\lambda(\Gamma)}\, d\Gamma < \infty$$

if we assume that $\int_K dk = 1$.

Let σ be a representation of G on a finite-dimensional Hilbert space V.

For any linear transformation T on V, put $\| T \| = \{\mathrm{sp}(T^*T)\}^{\frac{1}{2}}$, where T^* is the adjoint of T.

COROLLARY 2. *We can find an integer $m \geqq 0$ and a positive number c such that $\| \sigma(x) \| \leqq c \| x \|^m$ for all $x \in G$.*

Let us denote the corresponding representation of $\mathfrak{g}$ also by σ. If S is a nonsingular linear transformation on V, we know that $\| S\sigma(x)S^{-1} \| \leqq \| S \| \| \sigma(x) \| \| S^{-1} \|$, and therefore it is permissible, for our proof, to replace σ by an equivalent representation on V. Since $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}$, we may, by going over to an equivalent representation, assume that $\sigma(X)$ is skew-adjoint for $X \in \mathfrak{u}$. Then $\sigma(k)$ is unitary and $\sigma(a)$ self-adjoint ($k \in K, a \in A_{\mathfrak{p}}$). Therefore, since $G = KA_{\mathfrak{p}}{}^+K$, it is enough to consider the case when $x \in A_{\mathfrak{p}}{}^+$. Let $d = \dim V$ and $\Lambda_1, \cdots, \Lambda_r$ be all the distinct weight of σ (with respect to $\mathfrak{a}$). Then Λ_i are all real linear functions on $\mathfrak{a}$ and so from Lemma 6, we can choose an integer $m \geqq 0$ such that $e^{\Lambda_i(\Gamma)} \leqq \| \exp \Gamma \|^m$ $(1 \leqq i \leqq r)$ for $\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}{}^+$. This proves that

$$\| \sigma(a) \|^2 \leqq d \max_i e^{2\Lambda_i(\Gamma)} \leqq d \| a \|^{2m}$$

if $a = \exp \Gamma$ $(\Gamma \in \mathfrak{a}_{\mathfrak{p}_0}{}^+)$.

4. Applications of the above inequalities. Let $\mathcal{S}_0$ denote the set of all continuous functions f on $\mathfrak{g}_0$ such that

$$\nu_m(f) = \sup_{X \in \mathfrak{g}_0} |f(X)|(1 + \| X + \theta(X) \|)^m < \infty$$

for every integer $m \geqq 0$. Also let $\mathfrak{h}_0''$ denote the set of those points $H \in \mathfrak{h}_0$ where $\pi_0(H) \neq 0$.

LEMMA 7. *Let m be a given integer $\geqq 0$. Then we can find integers $q, m_1 \geqq 0$ and a positive number c, such that*

$$(1 + \| H \|)^{q_1} |\pi_0(H)|^q \int_G \| x \|^m |f(xH)| \, dx \leqq c\nu_{m_1+q_1}(f)$$

for $H \in \mathfrak{h}_0''$, $f \in \mathcal{S}_0$ and any integer $q_1 \geqq 0$. Moreover if Ω is a compact subset of $G \times \mathfrak{h}_0'' \times G$ and $f \in \mathcal{S}_0$, the integral

$$\int_G \| zxy \|^m |f(zxyH)| \, dx$$

converges uniformly with respect to $(z, H, y) \in \Omega$.

Choose an integer $s \geqq 0$ such that $\int_G \| x \|^{-s} dx < \infty$ (Corollary 1 to

Lemma 6). Then from Lemma 5, we can select integers $q, m_1 \geqq 0$ and a positive number c_1 with the property that

$$|\pi_0(H)|^q \| x \|^{m+s} \leqq c_1 \| xH + \theta(xH) \|^{m_1}$$

for all $H \in \mathfrak{h}_0$ and $x \in G$. Then since $\| H \| \leqq \| xH + \theta(xH) \|$ (Lemma 5),

$$(1 + \| H \|)^{q_1} |\pi_0(H)|^q \| x \|^m |f(xH)| \leqq c_1 \nu_{m_1+q_1}(f) \| x \|^{-s},$$

and therefore if we put

$$c = c_1 \int \| x \|^{-s} dx,$$

our first assertion follows. Now suppose $(z, H, y) \in \Omega$ and $f \in \mathcal{S}_0$. Then

$$\| zxy \|^m |f(zxyH)| \leqq c_1 |\pi_0(H)|^{-q} \nu_{m_1}(f) \| zxy \|^{-s}.$$

But $\| x \| = \| z^{-1}(zxy)y^{-1} \| \leqq \| z^{-1} \| \| zxy \| \| y^{-1} \|$. Therefore

$$\| zxy \| \geqq \| z^{-1} \|^{-1} \| x \| \| y^{-1} \|^{-1}$$

and

$$\| zxy \|^m |f(zxyH)| \leqq c_1 |\pi_0(H)|^{-q} \nu_{m_1}(f) \| z^{-1} \|^s \| y^{-1} \|^s \| x \|^{-s}$$

It is obvious that $|\pi_0(H)|^{-q} \| z^{-1} \|^s \| y^{-1} \|^s$ remains bounded on the compact set Ω, and so our statement about uniform convergence follows from the fact that $\int \| x \|^{-s} dx < \infty$.

From now on we shall make use of the notation and terminology of [2(h)] without further comment. Let $\mathcal{S}$ be the set of all $f \in C^\infty(\mathfrak{g}_0)$ such that $Df \in \mathcal{S}_0$ for every $D \in \partial(S(\mathfrak{g}))$. Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$, and define Γ_X $(X \in \mathfrak{g}_0)$ as in [2(h), §3]. Then we have seen (Lemma 3 of [2(h)]) that

$$g(x; b : X; \partial(\xi)) = g(x : X; \partial(\Gamma_X(b \times \xi))),$$

where $g \in C^\infty(\mathfrak{g}_0)$, $b \in \mathfrak{B}$, $\xi \in S(\mathfrak{g})$, $x \in G$ and $X \in \mathfrak{g}_0$. Suppose b and ξ are fixed. Then it is clear, from the definition of Γ_X, that we can select a finite number of linearly independent elements $\xi_1, \cdots, \xi_r \in S(\mathfrak{g})$ and also $p_1, \cdots, p_r \in S(\mathfrak{g})$ such that

$$\Gamma_X(b \times \xi) = p_1(X)\xi_1 + \cdots + p_r(X)\xi_r \qquad (X \in \mathfrak{g}_0).$$

Therefore

$$g(x; b : X; \partial(\xi)) = \sum_{1 \leqq i \leqq r} p_i(X) g(x : X; \partial(\xi_i)).$$

On the other hand, if $\eta \in S(\mathfrak{g})$,

$$g(x : X; \partial(\eta)) = g(xX; \partial(\eta^x)).$$

Therefore

$$g(x:X;\partial(\xi_i)) = g(xX;\partial(\xi_i^x)) \qquad (1 \leqq i \leqq r),$$

and again we can select a finite number of linearly independent elements η_j $(1 \leqq j \leqq s)$ in $S(\mathfrak{g})$ such that

$$\xi_i^x = \sum_{1 \leqq j \leqq s} a_{ij}(x)\eta_j \qquad (1 \leqq i \leqq r),$$

where a_{ij} are analytic functions on G. Moreover, we know from Corollary 2 to Lemma 6 that $|a_{ij}(x)| \leqq a \| x \|^m$ $(x \in G)$, where m is a suitable non-negative integer and a a positive constant. Then

$$|g(x;b:X;\partial(\xi))| \leqq a \| x \|^m \sum_{i,j} |p_i(X)|\,|g(xX;\partial(\eta_j))|.$$

Now suppose $g \in \mathcal{S}$. Then $\partial(\eta_j)g \in \mathcal{S}_0$, and it follows from Lemma 7, that

$$\int \| xy \|^m |g(xyH;\partial(\eta_j))|\,dx$$

converges uniformly with respect to (y, H) on every compact subset Ω of $G \times \mathfrak{h}_0''$. Thus we have obtained the following result.

LEMMA 8. *Suppose $f \in \mathcal{S}$, $b \in \mathfrak{B}$ and $\xi \in S(\mathfrak{g})$. Then if Ω is a compact subset of $G \times \mathfrak{h}_0''$, the integral*

$$\int_G |f(xy;b:H;\partial(\xi))|\,dx$$

converges uniformly with respect to $(y, H) \in \Omega$.

Put $\mathfrak{B}' = \mathfrak{B}\mathfrak{g}$.

COROLLARY 1. *The notation being as above,*

$$\int_G f(x;b:H;\partial(\xi))\,dx = 0$$

if $b \in \mathfrak{B}'$.

For a fixed $H \in \mathfrak{h}_0''$, put

$$F(y) = \int_G f(xy:H;\partial(\xi))\,dx \qquad (y \in G).$$

Then it follows from Lemma 8 that F is of class C^∞ on G and

$$F(y;b) = \int_G f(xy;b:H;\partial(\xi))\,dx$$

for $b \in \mathfrak{B}$. But since the above integral for $F(y)$ converges absolutely, it

follows from the right invariance of the Haar measure of G, that $F(y)$ is actually independent of y. Therefore $F(y;b)=0$ if $b\in\mathfrak{B}'$, and our assertion follows by putting $y=1$.

COROLLARY 2. *Put* $\phi_f'(H)=\int_G f(xH)\,dx$ *for* $f\in\mathcal{S}$ *and* $H\in\mathfrak{h}_0''$. *Then* ϕ_f' *is of class* C^∞ *on* $\mathfrak{h}_0''$ *and if* $v\in S(\mathfrak{h})$,

$$\phi_f'(H;\partial(v))=\int_G f(x:H;\partial(v))\,dx \qquad (H\in\mathfrak{h}_0'').$$

This is an immediate consequence of the uniform convergence of

$$\int_G |f(x:H;\partial(v))|\,dx \qquad (v\in S(\mathfrak{h}))$$

with respect to H on each compact subset of $\mathfrak{h}_0''$.

LEMMA 9. *For given* $b\in\mathfrak{B}$ *and* $\xi\in S(\mathfrak{g})$, *we can select integers* $q, M\geqq 0$ *and a finite number of elements* $d_1,\cdots,d_r\in\partial(S(\mathfrak{g}))$ *such that*

$$|\pi_0(H)|^q(1+\|H\|)^{q_1}\int |f(x;b:H;\partial(\xi))|\,dx\leqq\sum_{1\leqq i\leqq r}\nu_{M+q_1}(d_if) \qquad (H\in\mathfrak{h}_0'')$$

for $f\in\mathcal{S}$ *and any integer* $q_1\geqq 0$.

Let us keep to the notation of the proof of Lemma 8. Select q, m_1 and c corresponding to Lemma 7. Since p_i are polynomials, we can select an integer $q_2\geqq 0$ and a positive number c_1 such that

$$|p_i(X)|\leqq c_1(1+\|X\|)^{q_2} \qquad (1\leqq i\leqq r)$$

for all $X\in\mathfrak{g}$. Then if $q_1\geqq 0$, $f\in\mathcal{S}$ and $H\in\mathfrak{h}_0''$,

$$\begin{aligned}(1+\|H\|)^{q_1}|\pi_0(H)|^q\int &|f(x;b:H;\partial(\xi))|\,dx\\ &\leqq|\pi_0(H)|^q(1+\|H\|)^{q_1+q_2}ac_1r\sum_j\int\|x\|^m|f(xH;\partial(\eta_j))|\,dx\\ &\leqq acc_1r\sum_j\nu_{m_1+q_1+q_2}(\partial(\eta_j)f).\end{aligned}$$

The assertion of the lemma is now obvious.

Define ϕ_f' $(f\in\mathcal{S})$ as in Corollary 2 to Lemma 8.

COROLLARY. *For each* $d\in\mathfrak{D}(\mathfrak{h})$, *there exist integers* $q, m\geqq 0$ *and a finite number of elements* $d_1,\cdots,d_r\in\partial(S(\mathfrak{g}))$ *such that*

$$|\pi_0(H)|^q(1+\|H\|)^{q_1}|\phi_f'(H;d)|\leqq\sum_{1\leqq i\leqq r}\nu_{m+q_1}(d_if) \qquad (H\in\mathfrak{h}_0'')$$

for $f\in\mathcal{S}$ *and any integer* $q_1\geqq 0$.

It is obviously enough to prove this when d is of the form $u\partial(v)$ $(u, v \in S(\mathfrak{g}))$. Select an integer $q_2 \geqq 0$ and a positive number c such that $|u(H)| \leqq c(1 + \|H\|)^{q_2}$ $(H \in \mathfrak{h})$. Then

$$|\phi_f'(H;d)| \leqq c(1+\|H\|)^{q_2} \int |f(x:H;\partial(v))|\,dx$$

from Corollary 2 to Lemma 8, and the desired result now follows from Lemma 9.

We recall that $I(\mathfrak{g})$ is the algebra of those polynomials in $S(\mathfrak{g})$ which are invariant under G, and for any $p \in S(\mathfrak{g})$, $\tilde{p}$ denotes the restriction of p on $\mathfrak{h}$. Furthermore, $\pi = \prod_{\alpha \in P} \alpha$.

THEOREM 1. *For any $f \in \mathcal{S}$, put*

$$\phi_f(H) = \pi(H) \int_G f(xH)\,dx \qquad (H \in \mathfrak{h}_0'').$$

Then ϕ_f is a function of class C^∞ on $\mathfrak{h}_0''$ and

$$\phi_{\partial(\xi)f} = \partial(\tilde{\xi})\phi_f \qquad (\xi \in I(\mathfrak{g})).$$

The first statement follows from Corollary 2 to Lemma 8. Let $\mathfrak{h}_0'$ be the set of those points $H \in \mathfrak{h}_0$ where $\pi(H) \neq 0$. Then $\mathfrak{h}_0'$ is a dense subset of $\mathfrak{h}_0''$, and so for a fixed $\xi \in I(\mathfrak{g})$, it would be enough to prove that $\phi_{\partial(\xi)f} - \partial(\tilde{\xi})\phi_f$ is zero on $\mathfrak{h}_0'$. Fix an element $H \in \mathfrak{h}_0'$, and put $d = \partial(\tilde{\xi}) \circ \pi$. Then $d \in \mathfrak{D}(\mathfrak{h})$, and we know from Lemma 8 of [2(h)] that

$$f(xH;\partial(\xi)) = \pi(H)^{-1} f(x:H;d_H) + \sum_{1 \leqq i \leqq r} f(x;b_i:H;\partial(v_i)) \qquad (x \in G),$$

where $b_i \in \mathfrak{B}'$, $v_i \in S(\mathfrak{h})$ $(1 \leqq i \leqq r)$ and d_H is the local expression of d at H. Therefore it follows from Lemma 8 and its corollaries that

$$\begin{aligned}\phi_{\partial(\xi)f}(H) &= \pi(H) \int f(xH;\partial(\xi))\,dx = \int f(x:H;d_H)\,dx \\ &= \phi_f'(H;d) = \phi_f(H;\partial(\tilde{\xi})).\end{aligned}$$

This proves the theorem.

For any integer $m \geqq 0$ and $d \in \partial(S(\mathfrak{g}))$, consider the function $\nu_{m,d}$: $f \to \nu_m(df)$ $(f \in \mathcal{S})$ on $\mathcal{S}$. Obviously, it is a seminorm, and we can define a topology on $\mathcal{S}$ by means of the collection of all such seminorms $\nu_{m,d}$. In this way $\mathcal{S}$ becomes a locally convex space. Since $\mathfrak{h}_0''$ is an open subset of $\mathfrak{h}_0$, the space $\mathcal{E}(\mathfrak{h}_0'')$ can be defined as usual (see [2(h), §2]).

THEOREM 2. *For any f in $\mathcal{S}$, ϕ_f lies in $\mathcal{E}(\mathfrak{h}_0'')$ and $f \to \phi_f$ is a continuous mapping of $\mathcal{S}$ into $\mathcal{E}(\mathfrak{h}_0'')$.*

Let U be a subset of $\mathfrak{h}_0''$. We shall say that U has property (A) if for every $v \in S(\mathfrak{h})$, we can select an integer $M \geqq 0$, a real constant $c \geqq 1$ and a finite number of elements $d_1, \cdots, d_r \in \partial(S(\mathfrak{g}))$ such that

$$\text{(A)} \qquad (1 + \| H \|)^{q'} \, | \phi_f(H; \partial(v)) | \leqq c^{q'} \sum_{1 \leqq j \leqq r} {}_{M+q'}(d_j f)$$

for $H \in U$, $f \in \mathcal{S}$ and all integers $q' \geqq 0$. Clearly the union of a finite number of sets with property (A) again has the same property. Similarly (A) holds for the closure of U in $\mathfrak{h}_0''$ whenever it holds for U. Let w be the order of the Weyl group W (of $\mathfrak{g}$ with respect to $\mathfrak{h}$). From Lemma 11 of [2(h)], we can select homogeneous elements $v_1, \cdots, v_w \in S(\mathfrak{h})$, such that $S(\mathfrak{h}) = \sum_{1 \leqq i \leqq w} I(\mathfrak{h}) v_i$. For any $v \in S(\mathfrak{h})$, let $\gamma_i(v)$ denote the unique elements in $I(\mathfrak{g})$ such that $v = \sum_{1 \leqq i \leqq w} \overline{\gamma_i(v)} v_i$ (see [2(h), Lemmas 9 and 11]), and put $D_i(v) = \partial(\gamma_i(v))$. Then if $f \in \mathcal{S}$, we know from Theorem 1 that

$$\partial(v) \phi_f = \sum_{1 \leqq i \leqq w} \partial(v_i) \phi_{D_i(v) f},$$

and therefore

$$| \phi_f(H; \partial(v)) | \leqq \sum_{1 \leqq i \leqq w} | \phi_{D_i(v) f}(H; \partial(v_i)) | \qquad (H \in \mathfrak{h}_0'').$$

This shows that in order to prove that U has property (A), it is enough to check this property for $v = v_1, v_2, \cdots, v_w$ only. Finally, it is obvious that Theorem 2 would be proved if we can show that $\mathfrak{h}_0''$ has property (A).

Since $\mathfrak{h}_0'$ is a dense subset of $\mathfrak{h}_0''$ and it has only a finite number of connected components, it is enough to prove that each connected component of $\mathfrak{h}_0'$ has property (A). Let $\mathfrak{h}_+$ be a connected component of $\mathfrak{h}_0'$ and P' the set of those roots α (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) for which $-(-1)^{\frac{1}{2}}\alpha$ takes only positive values on $\mathfrak{h}_+$. Then we can select a fundamental system $(\alpha_1, \cdots, \alpha_l)$ of roots in P' (see [2(f), p. 749]). Choose a base $H_1, \cdots, H_l$ for $\mathfrak{h}_0$ over R such that $\alpha_i(H_j) = (-1)^{\frac{1}{2}}\delta_{ij}$ $(1 \leqq i, j \leqq l)$, and let $(t_1, \cdots, t_l)$ denote the system of Cartesian coordinates in $\mathfrak{h}_0$ corresponding to this base. By a monomial we shall mean a function T on $\mathfrak{h}_0$ of the form $T = t_1^{q_1} t_2^{q_2} \cdots t_l^{q_l}$, where $q_1, \cdots, q_l$ are nonnegative integers. q_i is called the exponent of t_i in T. In particular the monomial whose exponents are all zero is 1. We introduce a lexicographic order in the set of all monomials as follows. Suppose $T = t_1^{q_1} \cdots t_l^{q_l}$, $T' = t_1^{q_1'} \cdots t_l^{q_l'}$ are two distinct monomials. Then we say that $T \prec T'$ (or $T' \succ T$) if $q_i < q_i'$ where i is the least index $(1 \leqq i \leqq l)$ such that $q_i \neq q_i'$. It is obvious that every nonempty set of monomials has a lowest element. Furthermore, observe that every monomial takes only positive values on $\mathfrak{h}_+$. We say that a monomial T has property

(A′), if we can choose an integer $M \geqq 0$, a real number $c \geqq 1$ and a finite set of elements $d_1, \cdots, d_r \in \partial(S(\mathfrak{g}))$ such that

$$\text{(A′)} \qquad T(H)(1 + \| H \|)^{q'} \, | \phi_f(H; \partial(v_i)) | \leqq c^{q'} \sum_{1 \leqq j \leqq r} \nu_{M+q'}(d_j f) \qquad (1 \leqq i \leqq w)$$

for all $f \in \mathcal{S}$, $H \in \mathfrak{h}_+$ and $q' \geqq 0$. In order to prove that $\mathfrak{h}_+$ has property (A), it is enough to show that the monomial 1 has property (A′).

First of all we claim that monomials with property (A′) actually do exist. If $\alpha \in P'$, we know that $\alpha = \sum_{1 \leqq i \leqq l} m_i \alpha_i$, where the m_i are nonnegative integers. Select an index i such that $m_i \geqq 1$. Then it is clear that $| \alpha(H) | \geqq t_i(H)$ $(H \in \mathfrak{h}_+)$. This shows that there exists a monomial T_0 such that $| \pi_0(H) | \geqq T_0(H)$ for $H \in \mathfrak{h}_+$. Moreover, it follows easily from the Corollary to Lemma 9 that T_0^q has property (A′) for a suitable integer $q \geqq 0$.

Now let T be the lowest monomial with property (A′). We have to prove that $T = 1$. So let us assume that $T \neq 1$, and let m be the least index $(1 \leqq m \leqq l)$ such that the exponent q of t_m in T is positive. Put $t = t_m$. Then $T = t^q T_1$, where T_1 is a monomial in which the exponents of t_i are zero for $1 \leqq i \leqq m$. Write $\partial' = \partial(H_m) = \partial/\partial t$, and let

$$H_m v_i = \sum_{1 \leqq j \leqq w} p_{ij} v_j \qquad (1 \leqq i \leqq w).$$

where $p_{ij} \in I(\mathfrak{g})$. Fix an element $f \in \mathcal{S}$. Then

$$\partial'(\partial(v_i)\phi_f) = \sum_j \partial(v_j)\phi_{\partial(p_{ij})f}.$$

Set $\phi_i = T_1 \partial(v_i)\phi_f$ and $\psi_i = T_1 \sum_{1 \leqq j \leqq w} \partial(v_j)\phi_{\partial(p_{ij})f}$ $(1 \leqq i \leqq w)$. Then $\partial' \phi_i = \psi_i$. Moreover, if $M, c, d_1, \cdots, d_r$ are selected in accordance with (A′) for $T = t^q T_1$, it is clear that

$$t^q \mu^{q'} \, | \psi_i | \leqq c^{q'} a_{q'}, \; t^q \mu^{q'} \, | \phi_i | \leqq c^{q'} b_{q'} \qquad (1 \leqq i \leqq w, q' \geqq 0)$$

on $\mathfrak{h}_+$. Here $\mu(H) = 1 + \| H \|$ $(H \in \mathfrak{h}_0)$ and

$$a_{q'} = \sum_{1 \leqq j \leqq r} \sum_{1 \leqq i,k \leqq w} \nu_{M+q'}(d_j \partial(p_{ik}) f),$$

$$b_{q'} = \sum_{1 \leqq j \leqq r} \nu_{M+q'}(d_j f).$$

Let τ_+, τ_- denote the subsets of $\mathfrak{h}_+$ defined by the conditions $t \geqq 1$ and $0 < t \leqq 1$, respectively. For any $H \in \mathfrak{h}_+$, let H_0 denote the unique point of intersection of the line $H + sH_m$ $(s \in R)$ with the "hyperplane" $\sigma = \tau_+ \cap \tau_-$. Put $\tilde{\mu}(H) = \mu(H_0)$ $(H \in \mathfrak{h}_+)$. Then if $H \in \tau_-$,

$$\mu(H) \leqq 1 + \| H_0 \| + \| H_m \| \leqq (1 + \| H_0 \|) c_1 = c_1 \tilde{\mu}(H),$$

where $c_1 = 1 + \| H_m \|$. Similarly $\bar{\mu}(H) \leqq c_1 \mu(H)$. Now fix an integer $q' \geqq 0$. Then

$$| \psi_i | \leqq c^{q'} a_{q'} \mu^{-q'} t^{-q} \leqq (cc_1)^{q'} a_{q'} (\bar{\mu})^{-q'} t^{-q}$$

on τ_-. Suppose H is a point in τ_- and H_0 the corresponding point on σ. Then by integrating the differential equation $\partial' \phi_i = \psi_i$ along the line segment $t_k = t_k(H)$ $(k \neq m, 1 \leqq k \leqq l)$, $t(H) \leqq t = t_m \leqq 1$, we find that

$$| \phi_i(H_0) - \phi_i(H) | = \left| \int_{t(H)}^{1} \psi_i(H_0 - (1-s)H_m) ds \right| \leqq (cc_1)^{q'} a_{q'} (\bar{\mu}(H))^{-q'} \int_{t(H)}^{1} s^{-q} ds$$

since $\bar{\mu}$ is constant on this segment. Now put

$$\eta(H) = \begin{cases} \{t(H)^{-q+1} - 1\}/(q-1) & \text{if } q \geqq 2, \\ |\log t(H)| & \text{if } q = 1, \end{cases}$$

for $H \in \tau_-$. Then

$$| \phi_i(H) | \leqq | \phi_i(H_0) | + (cc_1)^{q'} a_{q'} (\bar{\mu}(H))^{-q'} \eta(H) \qquad (H \in \tau_-).$$

On the other hand, since $t(H_0) = 1$,

$$\mu(H_0)^{q'} | \phi_i(H_0) | \leqq c^{q'} b_{q'}.$$

Moreover, $\mu(H_0) = \bar{\mu}(H)$, and therefore

$$\begin{aligned} | \phi_i | &\leqq \{ c^{q'} b_{q'} + (cc_1)^{q'} a_{q'} \eta \} (\bar{\mu})^{-q'} \\ &\leqq \{ (cc_1)^{q'} b_{q'} + (cc_1^2)^{q'} a_{q'} \eta \} \mu^{-q'} \end{aligned}$$

on τ_-. Hence

$$t^{q-1} \mu^{q'} | \phi_i | \leqq (cc_1^2)^{q'} \{ b_{q'} + a_{q'} \eta t^{q-1} \}$$

on τ_-. On the other hand,

$$t^{q-1} \mu^{q'} | \phi_i | \leqq t^q \mu^{q'} | \phi_i | \leqq c^{q'} b_{q'} \leqq (cc_1^2)^{q'} b_{q'}$$

on τ_+. Now first suppose $q \geqq 2$. Then $\eta t^{q-1} \leqq 1$ on τ_-. Hence

$$t^{q-1} \mu^{q'} | \phi_i | \leqq (cc_1^2)^{q'} \{ b_{q'} + a_{q'} \} \qquad (1 \leqq i \leqq w)$$

on $\mathfrak{h}_+$. Then if we put $T_2 = t^{q-1} T_1$, the above inequality can be written as follows:

$$T_2 \mu^{q'} | \partial(v_i) \phi_f | \leqq (cc_1^2)^{q'} \sum_{1 \leqq j \leqq r} \{ \nu_{M+q'}(d_j f) + \sum_{1 \leqq h,k \leqq w} \nu_{M+q'}(d_j \partial(p_{hk}) f) \}$$

$(1 \leqq i \leqq w)$. Therefore since f and q' are arbitrary, it is clear that T_2 has property (A'). However, T_2 is lower than T, and so we get a contradiction with the definition of T. Hence we conclude that the case $q \geqq 2$ is impossible, and so $q = 1$. But then

$$\mu^{q'} | \phi_i | \leqq (cc_1^2)^{q'} \{ b_{q'} + a_{q'} | \log t | \}$$

on τ_-. Put $c_2 = cc_1^2$, $r' = r(w^2+1)$, and let d_j' $(1 \leqq j \leqq r')$ denote the r' differential operators $d_j\partial(p_{hk})$ and d_j $(1 \leqq j \leqq r, 1 \leqq h, k \leqq w)$. Then since f and q' were arbitrary, the above result shows that

$$T_1\mu^{q'} \mid \partial(v_i)\phi_g \mid \leqq c_2^{q'}\{\sum_{1\leqq j\leqq r'} \nu_{M+q'}(d_j'g)\}(1+\mid \log t \mid)$$

on τ_- for any $g \in \mathfrak{H}$ and $q' \geqq 0$. Applying this result to $\partial(p_{hk})f$, we obtain

$$\mu^{q'} \mid \psi_i \mid \leqq c_2^{q'}a'_{q'}(1+\mid \log t \mid) \qquad (1 \leqq i \leqq w, q' \geqq 0)$$

on τ_-, where

$$a'_{q'} = \sum_{1\leqq j\leqq r'} \sum_{1\leqq h,k\leqq w} \nu_{M+q'}(d_j'\partial(p_{hk})f).$$

Again, we find by integration that if $H \in \tau_-$,

$$\begin{aligned} \mid \phi_i(H) \mid &\leqq \mid \phi_i(H_0) \mid + \int_{t(H)}^{1} \mid \psi_i(H_0 - (1-s)H_m) \mid ds \\ &\leqq \{c^{q'}b_{q'} + 2(c_1c_2)^{q'}a'_{q'}\}(\mu(H))^{-q'}, \end{aligned}$$

since $\int_0^1 (1+\mid \log s \mid)ds = 2$. Therefore

$$\mu^{q'} \mid \phi_i \mid \leqq c_3^{q'}(b_{q'} + a'_{q'})$$

on τ_-, where $c_3 = 2c_1^2c_2$. On the other hand, since $q = 1$,

$$\mu^{q'} \mid \phi_i \mid \leqq t\mu^{q'} \mid \phi_i \mid \leqq c^{q'}b_{q'} \leqq c_3^{q'}b_{q'}$$

on τ_+. Therefore

$$\mu^{q'}T_1 \mid \partial(v_i)\phi_f \mid \leqq c_3^{q'}\{\sum_{1\leqq j\leqq r'} \sum_{1\leqq h,k\leqq w} \nu_{M+q'}(d_j'\partial(p_{hk})f) + \sum_{1\leqq j\leqq r} \nu_{M+q'}(d_jf)\}$$

on $\mathfrak{h}_+$ for $q' \geqq 0$ and $1 \leqq i \leqq w$. Again, since f is an arbitrary element of $\mathfrak{H}$ and c_3 is independent of f and q', it is clear that the monomial T_1 has property (A'), and so we get a contradiction as before. This completes the proof of Theorem 2.

5. The general case. We shall now discard the assumption that $\mathfrak{h}_0 \subset \mathfrak{k}_0$ and try to extend Theorems 1 and 2 of Section 4 to the general case. But first we need some preliminary work.

As before let $\bar{\theta}$ denote the conjugation of $\mathfrak{g}$ with respect to its compact real form $\mathfrak{u} = \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{p}_0$.

LEMMA 10. *Let* $\mathfrak{m}$ *be a subalgebra of* $\mathfrak{g}$ *such that* $\bar{\theta}(\mathfrak{m}) = \mathfrak{m}$. *Then* $\mathfrak{m}$ *is reductive in* $\mathfrak{g}$ [4].

It follows from our hypothesis that $\mathfrak{m}$ is the complexification of the

subalgebra $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{u}$. But $\mathfrak{u}$ being compact, $\mathfrak{m}_0$ is reductive in $\mathfrak{u}$ and so $\mathfrak{m}$ is reductive in $\mathfrak{g}$.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ such that $\theta(\mathfrak{h}_0) = \mathfrak{h}_0$. For any (real-valued) linear function λ on $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{p}_0$, let $\mathfrak{g}_{0\lambda}$ denote the space of those $X \in \mathfrak{g}_0$ for which $[H, X] = \lambda(H)X$ for all $H \in \mathfrak{h}_{\mathfrak{p}_0}$. Let $\mathfrak{m}_0$ be the orthogonal complement[6] of $\mathfrak{h}_{\mathfrak{p}_0}$ in $\mathfrak{g}_{00}$. Introduce an order [2(f), §2] in the space of linear functions on $\mathfrak{h}_{\mathfrak{p}_0}$ and put $\mathfrak{n}_0 = \sum_{\lambda>0} \mathfrak{g}_{0\lambda}$. Obviously, $\mathfrak{m}_0$ and $\mathfrak{n}_0$ are subalgebras of $\mathfrak{g}_0$ and $\mathfrak{n}_0$ is nilpotent. Moreover, it follows from its definition that $\theta(\mathfrak{m}_0) = \mathfrak{m}_0$. Therefore, from Lemma 10, $\mathfrak{m}_0$ is reductive in $\mathfrak{g}_0$.

Let G be a connected Lie group with the Lie algebra $\mathfrak{g}_0$, and let K, N and $A_{\mathfrak{p}}$ be the analytic subgroups of G corresponding to $\mathfrak{k}_0$, $\mathfrak{n}_0$ and $\mathfrak{h}_{\mathfrak{p}_0}$, respectively. Put $\mathfrak{m}_{\mathfrak{p}_0} = \mathfrak{m}_0 \cap \mathfrak{p}_0$.

LEMMA 11. *The mapping* $(k, X, a, n) \to k(\exp X)an$ $(k \in K, X \in \mathfrak{m}_{\mathfrak{p}_0}, a \in A_{\mathfrak{p}}, n \in N)$ *is a one-one regular analytic mapping of* $K \times \mathfrak{m}_{\mathfrak{p}_0} \times A_{\mathfrak{p}} \times N$ *onto* G.

Let M be the analytic subgroup of G corresponding to $\mathfrak{m}_0$. Since $\theta(\mathfrak{m}_0) = \mathfrak{m}_0$, it follows that $\mathfrak{m}_0 = \mathfrak{m}_0 \cap \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{p}_0}$. Moreover, $[\mathfrak{m}_0, \mathfrak{n}_0] \subset \mathfrak{n}_0$, and therefore $\mathfrak{s}_0 = \mathfrak{m}_0 + \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0$ is a Lie algebra and $\mathfrak{n}_0$ is an ideal in $\mathfrak{s}_0$. Hence $S = MA_{\mathfrak{p}}N$ is a group and N is normal in S. Let $M_{\mathfrak{p}}$ denote the subset of M consisting of elements of the form $\exp X$ $(X \in \mathfrak{m}_{\mathfrak{p}_0})$. First we shall show that $G = KM_{\mathfrak{p}}A_{\mathfrak{p}}N$. Extend $\mathfrak{h}_{\mathfrak{p}_0}$ to a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}_0}$ of $\mathfrak{p}_0$. Let $\tilde{\mathfrak{F}}$ and $\mathfrak{F}$ be the spaces of (real-valued) linear functions on $\mathfrak{a}_{\mathfrak{p}_0}$ and $\mathfrak{h}_{\mathfrak{p}_0}$, respectively. Select an order on $\tilde{\mathfrak{F}}$ compatible with the one already chosen on $\mathfrak{F}$ (see Section 2). Let $\tilde{\mathfrak{g}}_{0\lambda}$ $(\lambda \in \tilde{\mathfrak{F}})$ denote the set of those $X \in \mathfrak{g}_0$ for which $[H, X] = \lambda(H)X$ for all $H \in \mathfrak{a}_{\mathfrak{p}_0}$. Then if $\tilde{\mathfrak{n}}_0 = \sum_{\lambda>0} \tilde{\mathfrak{g}}_{0\lambda}$, it is clear that $\tilde{\mathfrak{n}}_0$ is a nilpotent subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{n}_0$. Let A, and $\tilde{N}$ be the analytic subgroups of G corresponding to $\mathfrak{a}_{\mathfrak{p}_0}$ and $\tilde{\mathfrak{n}}_0$, respectively. Then Iwasawa [3] has shown that $G = KA_{\cdot}\tilde{N}$. Moreover, in view of the above compatibility, it is clear that $\mathfrak{a}_{\mathfrak{p}_0} + \tilde{\mathfrak{n}}_0 \subset \mathfrak{g}_{00} + \mathfrak{n}_0 = \mathfrak{s}_0$. Hence $A_{\cdot}\tilde{N} \subset S$. Therefore $G = KA_{\cdot}\tilde{N} = KMA_{\mathfrak{p}}N$. Now we claim that $M = M_K M_{\mathfrak{p}}$, where $M_K = M \cap K$. This is seen as follows. Since $\mathfrak{h}_0$ is maximal abelian in $\mathfrak{g}_0$, it follows that $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{m}_0$. Therefore the center of $\mathfrak{m}_0$ is contained in $\mathfrak{h}_{\mathfrak{k}_0}$, and, since $\mathfrak{m}_0$ is reductive, $\mathfrak{m}_0' = [\mathfrak{m}_0, \mathfrak{m}_0]$ is the orthogonal complement of this center in $\mathfrak{m}_0$. This shows that $\mathfrak{m}_0' \supset \mathfrak{m}_{\mathfrak{p}_0}$.

[6] Whenever we speak of orthogonality or related concepts, they are to be understood with respect to the positive-definite Hermitian form $\| X \|^2$ $(X \in \mathfrak{g})$ on $\mathfrak{g}$.

Moreover $\mathfrak{m}_0$, and therefore also $\mathfrak{m}_0'$, is invariant under θ, so that $\mathfrak{m}_0' = \mathfrak{m}_0' \cap \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{p}_0}$. This means that $\mathfrak{m}_0' \cap \mathfrak{k}_0 + (-1)^{\frac{1}{2}}\mathfrak{m}_{\mathfrak{p}_0}$ is a semisimple subalgebra of $\mathfrak{u}$, and therefore it is compact. Then if M' and M_K', respectively, are the analytic subgroups of G corresponding to $\mathfrak{m}_0'$ and $\mathfrak{m}_0' \cap \mathfrak{k}_0$, we can conclude (see Mostow [5]) that $M' = M_K' M_{\mathfrak{p}}$. Now let $\mathfrak{c}_0$ be the center of $\mathfrak{m}_0$ and D the corresponding analytic subgroup of M. Then $\mathfrak{c}_0 \subset \mathfrak{m}_0 \cap \mathfrak{k}_0$, and therefore $D \subset M \cap K$ and $M = DM'$. This proves that $M = M_K M_{\mathfrak{p}}$, and hence $G = KM_{\mathfrak{p}}A_{\mathfrak{p}}N$.

Now in order to prove that our mapping is one-one, we shall first show that $M \cap A_{\mathfrak{p}}N = \{1\}$. Suppose $m = a \exp X$ $(m \in M, a \in A_{\mathfrak{p}}, X \in \mathfrak{n}_0)$. Then if[7] $b \in A_{\mathfrak{p}}$,

$$m = bmb^{-1} = a \exp \mathrm{Ad}(b)X.$$

Since the exponential mapping is well known to be one-one on $\tilde{\mathfrak{n}}_0$, it follows that $X = \mathrm{Ad}(b)X$ for all $b \in A_{\mathfrak{p}}$. This implies that $[H, X] = 0$ $(H \in \mathfrak{h}_{\mathfrak{p}_0})$, and hence $X \in \mathfrak{n}_0 \cap (\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_0) = \{0\}$. Therefore $m = a$. Let $\exp \mathfrak{p}_0$ denote the set of all elements of the form $\exp Y$ $(Y \in \mathfrak{p}_0)$. Then every element in G can be written *uniquely* in the form kp $(k \in K, p \in \exp \mathfrak{p}_0$, see Mostow [5]). On the other hand $M = M_K M_{\mathfrak{p}}$, and therefore if $m = kp$, we can conclude that $k \in M_K$, $p \in M_{\mathfrak{p}}$. Moreover, $a \in A_{\mathfrak{p}} \subset \exp \mathfrak{p}_0$. Therefore $k = 1$ and $p \in M_{\mathfrak{p}} \cap A_{\mathfrak{p}}$. But again, since the exponential mapping is one-one on $\mathfrak{p}_0$, $p = \exp Y$, where $Y \in \mathfrak{m}_{\mathfrak{p}_0} \cap \mathfrak{h}_{\mathfrak{p}_0} = \{0\}$. This proves that $m = 1$. Now suppose $k \in K \cap S$. Then $k = man$ $(m \in M, a \in A_{\mathfrak{p}}, n \in N)$, and $\mathrm{Ad}(k)$ leaves $\mathfrak{n}_0$ invariant. But $\mathrm{Ad}(k) = \theta \,\mathrm{Ad}(k)\theta^{-1}$, and therefore $\mathrm{Ad}(k)$ also leaves $\theta(\mathfrak{n}_0) = \sum_{\mu<0} \mathfrak{g}_{0\mu}$ invariant. This implies that $\mathrm{Ad}(n) = \mathrm{Ad}(a^{-1}m^{-1}k)$ leaves $\theta(\mathfrak{n}_0)$ invariant. However, this is possible only if $n = 1$ as follows from Lemma 13 below. Hence $k = ma$. Now let $m = m_1 p$, where $m_1 \in M_K, p \in M_{\mathfrak{p}}$. Since $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{m}_0$ commute, $m_1^{-1}k = pa \in K \cap (\exp \mathfrak{p}_0) = \{1\}$, and so $a = p^{-1}$. But $\mathfrak{m}_{\mathfrak{p}_0} \cap \mathfrak{h}_{\mathfrak{p}_0} = \{0\}$ and so $M_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \{1\}$. Therefore $a = p^{-1} = 1$, and this proves that $k = m_1 \in M_K$. Hence $K \cap S = M_K$. Let us now assume that $k_i m_i a_i n_i = k_2 m_2 a_2 n_2$ $(k_i \in K, m_i \in M_{\mathfrak{p}}, a_i \in A_{\mathfrak{p}}, n_i \in N, i = 1, 2)$. Put $m_i a_i n_i = s_i$. Then $k_1^{-1}k_2 \in K \cap S = M_K$, and therefore $k_2 = k_1 m$, where $m \in M_K$. Then $m_1^{-1}(mm_2) \in M \cap A_{\mathfrak{p}}N = \{1\}$. Hence $mm_2 = m_1$. But since both $m_1, m_2 \in M_{\mathfrak{p}} \subset \exp \mathfrak{p}_0$ and $m \in M_K \subset K$, it follows that $m = 1$ and $m_2 = m_1$. This proves that $k_1 = k_2$. Also one knows (see Iwasawa [3]) that the mapping $(a, n) \to an$ $(a \in A_+, n \in \bar{N})$ is one-one. Hence $a_1 = a_2$, $n_1 = n_2$, and this proves that the mapping of Lemma 11 is one-one.

[7] $x \to \mathrm{Ad}(x)$ $(x \in G)$ denotes, as usual, the adjoint representation of G.

The proof of the regularity is straightforward and we shall only sketch it here. Let ϕ denote our mapping. In order to prove that ϕ is regular at (k, X, a, n) $(k \in K, X \in \mathfrak{m}_{\mathfrak{p}_0}, a \in A_{\mathfrak{p}}, n \in N)$, we have to verify that the following linear mapping D of $\mathfrak{g}_0$ is nonsingular.[8]

$$DZ = Z \quad (Z \in \mathfrak{n}_0), \qquad DH = \mathrm{Ad}(n^{-1})H \quad (H \in \mathfrak{h}_{\mathfrak{p}_0}),$$

$$DY = \mathrm{Ad}(n^{-1}a^{-1})\{(1 - \exp(-\mathrm{ad}\, X))/\mathrm{ad}\, X\}Y \qquad (Y \in \mathfrak{m}_{\mathfrak{p}_0}),$$

$$DU = \mathrm{Ad}(n^{-1}a^{-1})\exp(-\mathrm{ad}\, X)U \qquad (U \in \mathfrak{k}_0).$$

Put $D' = \exp(\mathrm{ad}\, X)\mathrm{Ad}(an)D$. Then since $\mathfrak{g}_0$ is semisimple, it is clear that $\det D = \det D'$. Moreover, if I is the identity mapping of $\mathfrak{g}_0$, $\mathrm{Ad}(n) - I$ is nilpotent, and so the restriction of $\mathrm{Ad}(n)$ on $\mathfrak{n}_0$ has determinant 1. Similarly, since $X \in \mathfrak{m}_{\mathfrak{p}_0} \subset \mathfrak{m}_0'$, the restriction of $\exp(\mathrm{ad}\, X)$ on $\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0$ also has determinant 1. Hence if D_1 is the linear mapping of $\mathfrak{g}_0/(\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0)$ defined by D', $\det D = \det \mathrm{Ad}_{\mathfrak{n}_0}(a) \det D_1$, where $\mathrm{Ad}_{\mathfrak{n}_0}(a)$ is the restriction of $\mathrm{Ad}(a)$ on $\mathfrak{n}_0$. But D_1 leaves every element in $\mathfrak{k}_0 \bmod(\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0)$ fixed. Hence it defines a linear mapping D_2 of $\mathfrak{g}_0/(\mathfrak{k}_0 + \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0)$ and $\det D_1 = \det D_2$. On the other hand, it is clear that

$$\mathfrak{g}_0 = \mathfrak{g}_{00} + \sum_{\lambda > 0} \mathfrak{g}_{0\lambda} + \sum_{\mu < 0} \mathfrak{g}_{0\mu} = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_0 + \mathfrak{n}_0 + \theta(\mathfrak{n}_0).$$

Moreover, if $Z \in \mathfrak{n}_0$, then $\theta(Z) = (\theta(Z) + Z) - Z \in \mathfrak{k}_0 + \mathfrak{n}_0$, and so $\mathfrak{n}_0 + \theta(\mathfrak{n}_0) \subset \mathfrak{k}_0 + \mathfrak{n}_0$. Therefore $\mathfrak{g}_0 = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0} + \mathfrak{k}_0 + \mathfrak{n}_0$. We claim this sum is direct. For suppose $U \in \mathfrak{k}_0 \cap (\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0} + \mathfrak{n}_0)$. Then

$$U = \theta(U) \in (\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0} + \mathfrak{n}_0) \cap (\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0} + \theta(\mathfrak{n}_0)) = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0}.$$

Hence $U \in \mathfrak{k}_0 \cap (\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_{\mathfrak{p}_0}) \subset \mathfrak{k}_0 \cap \mathfrak{p}_0 = \{0\}$. Therefore we may identify $\mathfrak{g}_0/(\mathfrak{k}_0 + \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0)$ with $\mathfrak{m}_{\mathfrak{p}_0}$ in the obvious way. If $Y \in \mathfrak{m}_{\mathfrak{p}_0}$,

$$D'Y = \{(\exp(\mathrm{ad}\, X) - 1)/\mathrm{ad}\, X\}Y \equiv (\sinh \mathrm{ad}\, X/\mathrm{ad}\, X)Y \bmod \mathfrak{k}_0.$$

Therefore it is clear that

$$\det D = \det(\sinh \mathrm{ad}\, X/\mathrm{ad}\, X)_{\mathfrak{m}_{\mathfrak{p}_0}} \det \mathrm{Ad}_{\mathfrak{n}_0}(a)$$

(where the suffix $\mathfrak{m}_{\mathfrak{p}_0}$ denotes restriction on $\mathfrak{m}_{\mathfrak{p}_0}$). Since $\mathrm{Ad}(a)$ and $\sinh \mathrm{ad}\, X/\mathrm{ad}\, X$ are positive-definite self-adjoint operators on $\mathfrak{g}$, it follows that $\det D > 0$, and therefore ϕ is regular. In fact, the following lemma is an immediate consequence of the above computation.

[8] $\{(1 - \exp(-\mathrm{ad}\, X))/\mathrm{ad}\, X\}$ and $\sinh \mathrm{ad}\, X/\mathrm{ad}\, X$ are defined in the usual way by means of their convergent power series.

LEMMA 12. *Let dk, da, dn and dx denote the Haar measures on K, $A_{\mathfrak{p}}$, N and G, respectively, and dX the Euclidean measure on $\mathfrak{m}_{\mathfrak{p}_0}$. Then it is possible to normalize them in such a way that*

$$dx = \det(\sinh \operatorname{ad} X/\operatorname{ad} X)_{\mathfrak{m}_{\mathfrak{p}_0}} \det \operatorname{Ad}_{\mathfrak{n}_0}(a)\, dk da dn dX,$$

where $x = k(\exp X)an$ $(k \in K, X \in \mathfrak{m}_{\mathfrak{p}_0}, a \in A_{\mathfrak{p}_0}, n \in N)$.

The following corollary is now obvious from Lemmas 11 and 12.

COROLLARY. *The mapping $(k, X, n, a) \to k(\exp X)na$ is also a one-one regular mapping of $K \times \mathfrak{m}_{\mathfrak{p}_0} \times N \times A_{\mathfrak{p}}$ onto G, and if $x = k(\exp X)na$,*

$$dx = \det(\sinh \operatorname{ad} X/\operatorname{ad} X)_{\mathfrak{m}_{\mathfrak{p}_0}}\, dk dX dn da$$

with the same normalization of the measures as in Lemma 12.

In order to complete the proof of Lemma 11, we still have to prove the following result.

LEMMA 13. *Let X be an element in $\mathfrak{n}_0$ such that $\theta(\mathfrak{n}_0)$ is invariant under $\operatorname{Ad}(\exp X)$. Then $X = 0$.*

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ be all the positive elements $\lambda \in \mathfrak{F}$ such that $\mathfrak{g}_{0\lambda} \neq \{0\}$. Then if $X \neq 0$, let i be the largest index $(1 \leqq i \leqq r)$ such that $X \in \sum_{j \geqq i} \mathfrak{g}_{0\lambda_j}$. Then $X = X_i + X_{i+1} + \cdots + X_r$, where $X_j \in \mathfrak{g}_{0\lambda_j}$ $(1 \leqq j \leqq r)$ and $X_i \neq 0$. Now it is obvious that if $Y \in \mathfrak{g}_{0\lambda_i}$,

$$\exp(\operatorname{ad} X)\theta(Y) \equiv \theta(Y) + [X_i, \theta(Y)] \bmod \mathfrak{n}_0$$

and $[X_i, \theta(Y)] \in \mathfrak{g}_{00} = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_0$. However the left side lies in $\theta(\mathfrak{n}_0)$ by hypothesis. Therefore, since the sum $\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_0 + \mathfrak{n}_0 + \theta(\mathfrak{n}_0)$ is direct, we conclude that $[X_i, \theta(Y)] = 0$. Hence if $H \in \mathfrak{h}_{\mathfrak{p}_0}$,

$$\lambda_i(H)B(X_i, \theta(Y)) = B([H, X_i], \theta(Y)) = B(H, [X_i, \theta(Y)]) = 0.$$

Since $\lambda_i \neq 0$, we can select H such that $\lambda_i(H) \neq 0$, and thus obtain $B(X_i, \theta(Y)) = 0$. Now take $Y = X_i$. Then we get $\| X_i \|^2 = 0$, and so $X_i = 0$. This contradiction proves the lemma.

Select an order on the space of (real-valued) linear functions on $\mathfrak{h}_{\mathfrak{p}_0} + (-1)^{\frac{1}{2}}\mathfrak{h}_{\mathfrak{k}_0}$ compatible with the one already defined in $\mathfrak{F}$. Complexify $\mathfrak{h}_0$, $\mathfrak{m}_{\mathfrak{p}_0}$, $\mathfrak{n}_0$, etc., to $\mathfrak{h}$, $\mathfrak{m}_{\mathfrak{p}}$, $\mathfrak{n}$, respectively. Let P be the set of all positive roots (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) under our order. For each root α, select an element $X_\alpha \neq 0$ in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{h}$. Let P_+ be the set of those $\alpha \in P$ which are not identically zero on $\mathfrak{h}_{\mathfrak{p}}$. The remaining

roots in P are divided into two disjoint classes P_0 and P_- as follows. If α is a positive root which is not in P_+, then X_α lies either in $\mathfrak{p}$ or in $\mathfrak{k}$ (see [2(e), §2]), and, by definition, α is in P_0 or P_- accordingly. Then P is the disjoint union of P_+, P_0 and P_-, and it is obvious that $\mathfrak{n} = \sum_{\alpha \in P_+} CX_\alpha$. Define polynomial functions π_+, π_0 and π_- on $\mathfrak{h}$ by $\pi_+ = \prod_{\alpha \in P_+} \alpha$, $\pi_0 = \prod_{\beta \in P_0} \alpha$, $\pi_- = \prod_{\alpha \in P_-} \alpha$, and put $\pi = \pi_+ \pi_0 \pi_-$.

We shall now assume that G is the (connected) adjoint group of $\mathfrak{g}_0$. Let A be the Cartan subgroup [2(e), §2] of G corresponding to $\mathfrak{h}_0$. Then $A = A_K A_\mathfrak{p}$, where $A_K = A \cap K$ (Lemma 7 of [2(e)]). Let $x \to x^*$ denote the natural mapping of G on the factor space $G^* = G/A$. We put $x^* H = xH$ for $x \in G$ and $H \in \mathfrak{h}$. Since K is compact, we can normalize dk in such a way that $\int_K dk = 1$. Let dm denote the Haar measure on M and dx^* the invariant measure on G^*.

LEMMA 14. *It is possible to normalize dm and dx^* in such a way that*

$$\int_{G^*} f(x^*)\,dx^* = \int_{K \times M \times N} f(kmn)^*)\,dk\,dm\,dn$$

for $f \in C_c(G^*)$.

It is clear that if the Haar measure db on A is suitably normalized, then

$$\int g(x)\,dx = \int_{G^*} dx^* \int_A g(xb)\,db \qquad (g \in C_c(G)).$$

On the other hand, from the Corollary to Lemma 12,

$$\int g(x)\,dx = \int g(k \exp X n a)\, s(X)\,dk\,dX\,dn\,da,$$

where $s(X) = \det(\sinh \operatorname{ad} X / \operatorname{ad} X)_{\mathfrak{m}_\mathfrak{p}}$ $(X \in \mathfrak{m}_{\mathfrak{p}_0})$. Normalize the Haar measure dm' on $M_K = M \cap K$ in such a way that $\int dm' = 1$. Then it is well known (see [2(g), §12]) that dm can be so normalized that $dm = s(X)\,dm'\,dX$, where $m = m' \exp X$ $(m' \in M_K, X \in \mathfrak{m}_{\mathfrak{p}_0})$. Therefore it is obvious that

$$\int g(k \exp X n a)\, s(X)\,dk\,dX\,dn\,da = \int g(kmna)\,dk\,dm\,dn\,da.$$

On the other hand, $db = c\,da'\,da$ $(b = a'a, a' \in A_K, a \in A_\mathfrak{p})$, where c is a positive constant and da' is the Haar measure on A_K $(\int da' = 1)$. Therefore, since $a' M a'^{-1} = M$, $a' N a'^{-1} = N$ $(a' \in A_K)$, it is clear that

$$c\int g(kmna)\,dkdmdnda = \int_{K\times M\times N} dkdmdn \int_A g(kmnb)\,db.$$

Hence

$$c\int_{G^*} dx^* \int_A g(xb)\,db = \int_{K\times M\times N} dkdmdn \int_A g(kmnb)\,db$$

for $g\in C_c(G)$. Now suppose $f\in C_c(G^*)$. Then we can select (see Weil [7, p. 43]) $g\in C_c(G)$ such that

$$f(x^*) = \int_A g(xb)\,db \qquad (x\in G).$$

Hence

$$\int_{G^*} f(x^*)\,dx^* = c^{-1}\int_{K\times M\times N} f((kmn)^*)\,dkdmdn,$$

and this proves our assertion.

COROLLARY. *With the normalizations of Lemma* 14 *we also have*

$$\int_{G^*} f(x^*)\,dx^* = \int_{K\times M\times N} f((knm)^*)\,dkdndm$$

for $f\in C_c(G^*)$.

Since $[\mathfrak{m}_0, \mathfrak{n}_0]\subset \mathfrak{n}_0$, it follows that $mNm^{-1} = N$ $(m\in M)$. Therefore, in order to prove the corollary, it is sufficient to show that $|\det \mathrm{Ad}_{\mathfrak{n}}(m)| = 1$ $(m\in M)$, where $\mathrm{Ad}_{\mathfrak{n}}(m)$ is the restriction of $\mathrm{Ad}(m)$ on $\mathfrak{n}$. Let $m = m_1m_2$ $(m_1\in M_K, m_2\in M_{\mathfrak{p}})$. Since M_K is compact and $\mathfrak{m}_{\mathfrak{p}_0}\subset[\mathfrak{m}_0,\mathfrak{m}_0]$, it follows that $|\det \mathrm{Ad}_{\mathfrak{n}}(m_1)| = \det \mathrm{Ad}_{\mathfrak{n}}(m_2) = 1$, and therefore

$$|\det \mathrm{Ad}_{\mathfrak{n}}(m)| = |\det \mathrm{Ad}_{\mathfrak{n}}(m_1)\det \mathrm{Ad}_{\mathfrak{n}}(m_2)| = 1.$$

Put $\mathfrak{l}_0 = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{m}_0$. Then $\mathfrak{l}_0$ is a Lie algebra which, by Lemma 10, is reductive in $\mathfrak{g}_0$. Let $\mathfrak{l}$ denote the complexification of $\mathfrak{l}_0$ in $\mathfrak{g}$ and $\zeta(L)$ $(L\in\mathfrak{l})$ the determinant of the restriction of $\mathrm{ad}\,L$ on $\mathfrak{n}$. It is obvious that ζ is a polynomial function on $\mathfrak{l}$ which takes only real values on $\mathfrak{l}_0$.

LEMMA 15. *It is possible to normalize the Euclidean measure* dZ *on* $\mathfrak{n}_0$ *so that the following condition is fulfilled. If* L *is an element in* $\mathfrak{l}_0$ *such that* $\zeta(L)\neq 0$, *then*

$$\int_N f(nL)\,dn = |\zeta(L)|^{-1}\int_{\mathfrak{n}_0} f(L+Z)\,dZ \qquad (f\in C_c(\mathfrak{l}_0+\mathfrak{n}_0)).$$

COROLLARY. *Let* H *be an element in* $\mathfrak{h}_0$ *such that* $\pi_+(H)\neq 0$. *Then*

$$\int |f(x^*H)|\,dx^* = |\zeta(H)|^{-1}\int |f(k(mH+Z))|\,dkdmdZ$$

for $f\in C_c(\mathfrak{g}_0)$.

It is clear that $\zeta(mH) = \zeta(H) = \pi_+(H)$ $(m \in M)$. Our corollary is therefore an immediate consequence of Lemma 15 and the Corollary of Lemma 14. On the other hand, since $[\mathfrak{l}_0, \mathfrak{n}_0] \subset \mathfrak{n}_0$, it is clear that $nL - L \in \mathfrak{n}_0$ for $n \in N$ and $L \in \mathfrak{l}_0$. Therefore Lemma 15 follows from the result below.

LEMMA 16. *Let L be an element in $\mathfrak{l}_0$ such that $\zeta(L) \neq 0$. Then the mapping $Z \to \exp(\operatorname{ad} Z)L - L$ $(Z \in \mathfrak{n}_0)$ is a one-one regular mapping of $\mathfrak{n}_0$ onto itself.*

Let ϕ denote this mapping. Then in order to show that ϕ is regular at a point $Z \in \mathfrak{n}_0$, we have to consider the endomorphism D of $\mathfrak{n}_0$ given by

$$DY = -(\operatorname{ad} L)\{(1 - \exp(-\operatorname{ad} Z))/\operatorname{ad} Z\}Y \qquad (Y \in \mathfrak{n}_0)$$

and prove that $\det D \neq 0$. But since $\operatorname{ad} Z$ is nilpotent, it is clear that

$$|\det D| = |\zeta(L)| \neq 0,$$

and so ϕ is everywhere regular. Moreover, it is well known that $Z \to \exp Z$ is a one-one mapping of $\mathfrak{n}_0$ onto N. Therefore if $\exp(\operatorname{ad} Z)L = L$ $(Z \in \mathfrak{n}_0)$, it follows that $\exp(t \operatorname{ad} L)Z = Z$ $(t \in R)$, and therefore $[L, Z] = 0$. But since $\zeta(L) \neq 0$, this implies that $Z = 0$. This proves that $nL \neq L$ $(n \in N)$ unless $n = 1$, and therefore $\phi(Z_1) = \phi(Z_2)$ $(Z_1, Z_2 \in \mathfrak{n}_0)$ unless $Z_1 = Z_2$. Hence ϕ is univalent.

We have still to prove that $\phi(\mathfrak{n}_0) = \mathfrak{n}_0$. Since $\mathfrak{n}_0$ is connected, it would be enough to show that $\phi(\mathfrak{n}_0)$ is closed in $\mathfrak{n}_0$. This however is an immediate consequence of the following statement. *Suppose Z varies in $\mathfrak{n}$ in such a way that $\|\exp(\operatorname{ad} Z)L - L\|$ remains bounded. Then $\|Z\|$ also remains bounded.* This is proved as follows. Since $CL + Z$ is a solvable Lie algebra over the complex field, we can choose a base $Z_1, \cdots, Z_r$ for $\mathfrak{n}$ over C and complex numbers $c_1, \cdots, c_r$ such that $[L, Z_i] \equiv c_i Z_i \bmod \sum_{k>i} CZ_k$ and $[Z_i, Z_j] \in \sum_{k>j} CZ_k$ $(1 \leqq i \leqq j \leqq r)$. Moreover $c_i \neq 0$ since $\zeta(L) \neq 0$. Let $t_1, \cdots, t_r$ denote the complex Cartesian coordinates in $\mathfrak{n}$ corresponding to this base. Then it is clear that for any integer j $(1 \leqq j \leqq r)$, there exists a polynomial p_j in $(j-1)$ variables $\tau_1, \cdots, \tau_{j-1}$ with complex coefficients such that

$$t_j((\exp \operatorname{ad} Z)L - L) = -c_j t_j(Z) + p_j(t_1(Z), \cdots, t_{j-1}(Z))$$

for $Z \in \mathfrak{n}$. Hence it follows by induction on j that if $\|(\exp \operatorname{ad} Z)L - L\|$ remains bounded, the same holds for $|t_j(Z)|$. This proves our statement and therefore also the lemma.

In view of the fact that $|\det D| = |\zeta(L)|$, Lemma 15 is now obvious

Let $\mathfrak{q}_0$ be any subalgebra of $\mathfrak{g}_0$ such that $\theta(\mathfrak{q}_0) = \mathfrak{q}_0$, and let $\mathfrak{q}$ be the complexification of $\mathfrak{q}_0$ in $\mathfrak{g}$. It is obvious that the fundamental bilinear form B is nondegenerate on $\mathfrak{q}$, and we know that $\mathfrak{q}$ is reductive in $\mathfrak{g}$ (Lemma 10). For any $p \in S(\mathfrak{g})$, let $p_{\mathfrak{q}}$ denote the restriction of p on $\mathfrak{q}$. Moreover, if $\mathfrak{q}_0'$ is a nonempty open subset of $\mathfrak{q}_0$, consider the space $\mathcal{S}(\mathfrak{q}_0')$ of functions $f \in C^\infty(\mathfrak{q}_0')$ such that

$$\nu_m(df; \mathfrak{q}_0') = \sup_{X \in \mathfrak{q}_0'} |f(X; d)|(1 + \|X + \theta(X)\|)^m < \infty$$

for every $d \in \partial(S(\mathfrak{q}))$ and every integer $m \geqq 0$. We define a topology in $\mathcal{S}(\mathfrak{q}_0')$ by means of these seminorms $f \to \nu_m(df; \mathfrak{q}_0')$ $(m \geqq 0, d \in \partial(S(\mathfrak{q})))$. In this way, $\mathcal{S}(\mathfrak{q}_0')$ becomes a locally convex space. Similarly, put $\tau_m(df; \mathfrak{q}_0') = \sup_{X \in \mathfrak{q}_0'} |f(X; d)|(1 + \|X\|)^m$ for $d \in \mathfrak{D}(\mathfrak{q})$ and $f \in \mathcal{E}(\mathfrak{q}_0')$. We note that

$$\nu_m(df; \mathfrak{q}_0') \leqq 2^m \tau_m(df; \mathfrak{q}_0') \qquad (d \in \partial(S(\mathfrak{q}))$$

since $\|X + \theta(X)\| \leqq \|X\| + \|\theta(X)\| = 2\|X\|$ $(X \in \mathfrak{g}_0)$.

LEMMA 17. *For any $f \in \mathcal{S}(\mathfrak{g}_0)$, let g_f denote the function on I_0 given by*

$$g_f(L) = \int_{\mathfrak{n}_0} f(L + Z)\, dZ \qquad (L \in \mathrm{I}_0).$$

Then $g_f \in \mathcal{S}(\mathrm{I}_0)$ and $f \to g_f$ is a continuous mapping of $\mathcal{S}(\mathfrak{g}_0)$ into $\mathcal{S}(\mathrm{I}_0)$. Moreover $g_{\partial(p)f} = \partial(p_{\mathfrak{l}}) g_f$ for $p \in I(\mathfrak{g})$. Finally, if $f \in \mathcal{E}(\mathfrak{g}_0)$ then $g_f \in \mathcal{E}(\mathrm{I}_0)$ and $f \to g_f$ $(f \in \mathcal{E}(\mathfrak{g}_0))$ is a continuous mapping of $\mathcal{E}(\mathfrak{g}_0)$ into $\mathcal{E}(\mathrm{I}_0)$.

Notice that $\mathfrak{l}$, $\mathfrak{n}$ and $\theta(\mathfrak{n})$ are mutually orthogonal subspaces of $\mathfrak{g}$. Hence $\|L + Z + \theta(L + Z)\|^2 = \|L + \theta(L)\|^2 + 2\|Z\|^2$ $(L \in \mathrm{I}_0, Z \in \mathfrak{n}_0)$. Hence if $d \in \partial(S(\mathfrak{l}))$,

$$\begin{aligned}(1 + \|Z\|)^m |f(L + Z; d)| &\leqq (1 + \|L + Z + \theta(L + Z)\|)^m |f(L + Z; d)| \\ &\leqq \nu_m(df; \mathfrak{g}_0).\end{aligned}$$

We can obviously choose m so large that $\int (1 + \|Z\|)^{-m} dZ < \infty$. This shows that

$$\int_{\mathfrak{n}_0} |f(L + Z; d)|\, dZ \leqq \nu_m(df; \mathfrak{g}_0) \int (1 + \|Z\|)^{-m} dZ.$$

Hence the integral

$$\int_{\mathfrak{n}_0} f(L + Z; d)\, dZ$$

converges uniformly with respect to $L \in \mathfrak{l}_0$, and therefore g_f is of class C^∞ and

$$g_f(L;d) = \int_{\mathfrak{n}_0} f(L+Z;d)\,dZ.$$

Moreover, it is obvious that

$$\nu_{m_1}(dg_f;\mathfrak{l}_0) \leqq \nu_{m+m_1}(df;\mathfrak{g}_0) \int_{\mathfrak{n}_0} (1+\|Z\|)^{-m_1} dZ \qquad (m_1 \geqq 0).$$

Therefore $g_f \in \mathcal{S}(\mathfrak{l}_0)$ and the mapping $f \to g_f$ of $\mathcal{S}(\mathfrak{g}_0)$ into $\mathcal{S}(\mathfrak{l}_0)$ is continuous.

Now let f be a fixed element in $\mathcal{S}(\mathfrak{g}_0)$ and Ω a compact subset in $\mathfrak{g}_0$. For any $d \in \partial(S(\mathfrak{g}))$ and $Y \in \Omega$, consider the integral,

$$\int_{\mathfrak{n}_0} |f(Y+Z;d)|\,dZ.$$

Then

$$\|Z\| \leqq \|Z+\theta(Z)\| \leqq \|Z+Y+\theta(Z+Y)\| + \|Y+\theta(Y)\| \leqq \|X+\theta(X)\| + 2a \qquad (Y \in \Omega),$$

where $X = Y+Z$ and $a = \sup_{Y\in\Omega} \|Y\|$. Hence

$$(1+\|Z\|)^m |f(Y+Z;d)| \leqq (1+2a)^m \nu_m(df;\mathfrak{g}_0),$$

and this proves that the above integral converges uniformly with respect to $Y \in \Omega$. Now put

$$g(Y) = \int_{\mathfrak{n}_0} f(Y+Z)\,dZ \qquad (Y \in \mathfrak{g}_0).$$

Then it follows from the above result that $g \in C^\infty(\mathfrak{g}_0)$ and

$$g(Y;d) = \int_{\mathfrak{n}_0} f(Y+Z;d)\,dZ \qquad (d \in \partial(S(\mathfrak{g})), Y \in \mathfrak{g}_0).$$

But it is clear that $g(Y+Z) = g(Y)$ $(Z \in \mathfrak{n}_0)$. Therefore $dg = 0$ for $d \in \partial(S(\mathfrak{g})\mathfrak{n})$. Therefore, in order to prove the second statement of the lemma, it is enough to show that $p - p_{\mathfrak{l}} \in S(\mathfrak{g})\mathfrak{n}$ for $p \in I(\mathfrak{g})$. This will be done in Lemma 18 below.

Now suppose $f \in \mathcal{B}(\mathfrak{g}_0)$. Then since $\max(\|L\|, \|Z\|) \leqq \|L+Z\|$ $(L \in \mathfrak{l}_0, Z \in \mathfrak{n}_0)$,

$$(1+\|Z\|)^m (1+\|L\|)^{m_1} |f(L+Z;d)| \leqq \tau_{m+m_1}(df;\mathfrak{g}_0)$$

for $d \in \partial(S(\mathfrak{l}))$ and $m, m_1 \geqq 0$. Hence

$$\tau_{m_1}(dg_f;\mathfrak{l}_0) \leqq \tau_{m+m_1}(df;\mathfrak{g}_0) \int_{\mathfrak{n}_0} (1+\|Z\|)^{-m} dZ,$$

and this proves that $g_f \in \mathcal{C}(\mathfrak{l}_0)$ and $f \to g_f$ is a continuous mapping of $\mathcal{C}(\mathfrak{g}_0)$ into $\mathcal{C}(\mathfrak{l}_0)$.

For any $X \in \mathfrak{g}$, let d_X denote the derivation of $S(\mathfrak{g})$ which coincides with $\operatorname{ad} X$ on $\mathfrak{g}$.

LEMMA 18. *Let p be an element in $S(\mathfrak{g})$ such that $d_H p = 0$ for $H \in \mathfrak{h}_\mathfrak{p}$. Then $p - p_\mathfrak{l} \in S(\mathfrak{g})\mathfrak{n}$.*

Let $\alpha_1, \cdots, \alpha_r$ be all the distinct roots in P_+. Then $X_{\alpha_1}, \cdots, X_{\alpha_r}$ is a base for $\mathfrak{n}$ over C. Let $\mathcal{M}$ be the set of all monomials (in $S(\mathfrak{g})$) of the form $X_{\alpha_1}^{m_1} \cdots X_{\alpha_r}^{m_r}$, where $m_1, \cdots, m_r$ are nonnegative integers. For any monomial $M = X_{\alpha_1}^{m_1} \cdots X_{\alpha_r}^{m_r}$, put $\beta_M = m_1\alpha_1 + \cdots + m_r\alpha_r$ and extend θ to an automorphism of $S(\mathfrak{g})$. Then it is clear that $d_H\theta(M) = -\beta_M(H)\theta(M)$ $(H \in \mathfrak{h}_\mathfrak{p})$. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{l}$, $\mathfrak{n}$ and $\theta(\mathfrak{n})$, it follows that $S(\mathfrak{g})$ is the direct sum of $S(\mathfrak{l})S(\theta(\mathfrak{n}))$ and $S(\mathfrak{g})\mathfrak{n}$. Hence for any $q \in S(\mathfrak{g})$, there exist unique elements $q_M \in S(\mathfrak{l})$ $(M \in \mathcal{M})$ such that

$$q - \sum_M q_M\theta(M) \in S(\mathfrak{g})\mathfrak{n}.$$

Now apply d_H $(H \in \mathfrak{h}_\mathfrak{p})$ to this relation. Then

$$d_H q + \sum_M q_M\beta_M(H)\theta(M) \in S(\mathfrak{g})\mathfrak{n}$$

since $[\mathfrak{h}_\mathfrak{p}, \mathfrak{l}] = \{0\}$. In particular, if $d_H q = 0$, we get

$$\sum_M q_M\beta_M(H)\theta(M) \in S(\mathfrak{g})\mathfrak{n}.$$

But as we have observed above, $(S(\mathfrak{l})S(\theta(\mathfrak{n}))) \cap (S(\mathfrak{g})\mathfrak{n}) = \{0\}$. Therefore we conclude that $q_M\beta_M(H) = 0$ for every $M \in \mathcal{M}$. On the other hand, if $M \neq 1$, it is obvious that the restriction of β_M on $\mathfrak{h}_\mathfrak{p}$ is positive in our order, and so we can select $H \in \mathfrak{h}_\mathfrak{p}$ such that $\beta_M(H) \neq 0$. Therefore if $q = p$, we conclude that $p_M = 0$ for $M \neq 1$. This means that $p - p_1 \in S(\mathfrak{g})\mathfrak{n}$. But since $p_1 \in S(\mathfrak{l})$ and $\mathfrak{l}$ and $\mathfrak{n} + \theta(\mathfrak{n})$ are mutually orthogonal, it follows that $p_\mathfrak{l} = (p_1)_\mathfrak{l} = p_1$. Hence $p - p_\mathfrak{l} \in S(\mathfrak{g})\mathfrak{n}$.

COROLLARY. *Let p be an element in $S(\mathfrak{g})$ such that $d_H p = 0$ for all $H \in \mathfrak{h}_\mathfrak{p}$. Then $\partial(p_\mathfrak{l})g_f = g_{\partial(p)f}$ $(f \in \mathcal{C}(\mathfrak{g}_0))$ in the notation of Lemma 17.*

This follows immediately from Lemma 18.

Put $\pi_\mathfrak{m} = \pi_0\pi_-$ and let $\mathfrak{h}_0'$ be the set of all points $H \in \mathfrak{h}_0$ where $\pi(H) \neq 0$. We know that $\mathfrak{l}_0 = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0 + \mathfrak{m}_0'$, where $\mathfrak{c}_0$ is the center and $\mathfrak{m}_0'$ the derived algebra of $\mathfrak{m}_0$. Moreover, $\mathfrak{m}_0'$ is semisimple and $\mathfrak{m}_0' = \mathfrak{m}_0' \cap \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{p}_0}$. Finally, $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{m}_0$, and therefore $\mathfrak{h}_{\mathfrak{k}_0} \cap \mathfrak{m}_0'$ is a Cartan subalgebra of $\mathfrak{m}_0'$. Let D and M' be the analytic subgroups of M

corresponding to $\mathfrak{c}_0$ and $\mathfrak{m}_0'$. Then $D \subset K$ and $M = DM'$. Let $d\gamma$, dm' denote the Haar measures on D and M', respectively. We normalize them in such a way that $\int_D d\gamma = 1$ and

$$\int_M f(m)\,dm = \int_{M' \times D} f(m'\gamma)\,dm'\,d\gamma$$

for $f \in C_c(M)$. Now if $X \in \mathfrak{c}_0$ and $Y, Z \in \mathfrak{m}_0$, then $B(X,[Y,Z]) = B([X,Y],Z) = 0$, and from this it follows easily that $\mathfrak{c}_0$ and $\mathfrak{m}_0'$ are mutually orthogonal.[9] Moreover $\mathfrak{m}_0$ is orthogonal to $\mathfrak{h}_{\mathfrak{p}_0}$ by definition. Hence $\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0$ is orthogonal to $\mathfrak{m}_0'$. It is clear that $X_\alpha \in \mathfrak{m}$ for $\alpha \in P_0 \cup P_-$. Therefore since $[\mathfrak{h}_\mathfrak{p} + \mathfrak{c}, \mathfrak{m}] = \{0\}$, we conclude that α vanishes identically on $\mathfrak{h}_\mathfrak{p} + \mathfrak{c}$. This proves that $\pi_\mathfrak{m}(H) = \pi_\mathfrak{m}(H_-)$ $(H \in \mathfrak{h})$, where H_- is the orthogonal projection of H in $\mathfrak{h} \cap \mathfrak{m}'$.

For any $g \in \mathcal{S}(\mathrm{I}_0)$ and $X \in \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0$, let g_X denote the function on $\mathfrak{m}_0'$ given by $g_X(Y) = g(X+Y)$ $(Y \in \mathfrak{m}_0')$. Then if $d \in \partial(S(\mathfrak{m}'))$, it is clear that $(dg)_X = dg_X$. Moreover since $\mathfrak{m}_0'$ and $\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0$ are mutually orthogonal, we have

$$\nu_q(dg_X; \mathfrak{m}_0') \leqq \nu_q(dg; \mathrm{I}_0) \qquad (q \geqq 0).$$

This proves that $g_X \in \mathcal{S}(\mathfrak{m}_0')$. Furthermore,

$$(1 + \| X + \theta(X) \|)^{q_1}(1 + \| Y + \theta(Y) \|)^{q_2} \leqq (1 + \| X + Y + \theta(X+Y) \|)^{q_1+q_2}$$

$(X \in \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0;\ Y \in \mathfrak{m}_0';\ q, q_2 \geqq 0)$. Therefore

$$\sup_{X \in \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0} (1 + \| X + \theta(X) \|)^{q_1} \nu_{q_2}(dg_X; \mathfrak{m}_0') \leqq \nu_{q_1+q_2}(dg; \mathrm{I}_0).$$

Let $\mathfrak{h}_0''$ be the set of those points $H \in \mathfrak{h}_0$ where $\pi_0(H) \neq 0$. Then for $H \in \mathfrak{h}_0''$ and $g \in \mathcal{S}(\mathrm{I}_0)$, consider the integral

$$\pi_\mathfrak{m}(H) \int_M g(mH)\,dm = \pi_\mathfrak{m}(H_-) \int_{M'} g_{H_0}(m'H_-)\,dm',$$

where H_0 and H_-, respectively, are the orthogonal projections of H in $\mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{c}_0$ and $\mathfrak{h}_0 \cap \mathfrak{m}_0'$. The results of Section 4 are applicable to $\mathfrak{m}_0'$ since $\mathfrak{h}_{\mathfrak{k}_0} \cap \mathfrak{m}_0'$ is maximal abelian in $\mathfrak{m}_0'$. Therefore since $g_{H_0} \in \mathcal{S}(\mathfrak{m}_0')$ and $\pi_0(H_-) = \pi_0(H) \neq 0$, the above integral is well defined. We denote its value by $\psi_g(H)$. It follows easily from Lemma 9 (applied to $\mathfrak{m}_0'$) that ψ_g is of class C^∞ on $\mathfrak{h}_0''$ and $\psi_g(H;d) = \psi_{dg}(H)$ for $d \in \partial(S(\mathfrak{h}_\mathfrak{p} + \mathfrak{c}))$. Now suppose $d_1 \in \partial(S(\mathfrak{h}_\mathfrak{p} + \mathfrak{c}))$ and $d_2 \in \partial(S(\mathfrak{h} \cap \mathfrak{m}'))$. Then $\psi_g(H; d_1 d_2) = \psi_{d_1 g}(H; d_2)$. Moreover, as we have seen during the proof of Theorem 2, we can choose an integer $N \geqq 0$,

[9] Cf. footnote 6.

a real constant $c \geqq 1$ and a finite number of elements $d_1', \cdots, d_r' \in \partial(S(\mathfrak{m}'))$ such that

$$(1 + \| H_- \|)^{q'} |\psi_f(H; d_2)| \leqq c^{q'} \sum_{1 \leqq j \leqq r} \nu_{N+q'}(d_j' f_{H_0}; \mathfrak{m}_0')$$

for any $f \in \mathcal{S}(\mathrm{I}_0)$, $H \in \mathfrak{h}_0''$ and $q' \geqq 0$. Therefore

$$\begin{aligned}(1 + \| H + \theta(H) \|)^{q'} &|\psi_g(H; d_1 d_2)| \\ &\leqq (1 + \| H_0 + \theta(H_0) \|)^{q'} (1 + 2 \| H_- \|)^{q'} |\psi_{d_1 g}(H; d_2)| \\ &\leqq (2c)^{q'} (1 + \| H_0 + \theta(H_0) \|)^{q'} \sum_{1 \leqq j \leqq r} \nu_{N+q'}(d_j' (d_1 g)_{H_0}; \mathfrak{m}_0') \\ &\leqq (2c)^{q'} \sum_{1 \leqq j \leqq r} \nu_{N+2q'}(d_j' d_1 g; \mathrm{I}_0)\end{aligned}$$

since $d_j'(d_1 g)_{H_0} = (d_j' d_1 g)_{H_0}$. This proves that $\psi_g \in \mathcal{S}(\mathfrak{h}_0'')$ and the mapping $g \to \psi_g$ of $\mathcal{S}(\mathrm{I}_0)$ into $\mathcal{S}(\mathfrak{h}_0'')$ is continuous. Let $I(\mathrm{I})$ denote the set of all elements in $S(\mathrm{I})$ which are invariant under M. Then it follows from Theorem 1 (applied to $\mathfrak{m}_0'$) that $\psi_{\partial(p)g} = \partial(\tilde{p})\psi_g$ $(p \in I(\mathrm{I}), g \in \mathcal{S}(\mathrm{I}_0))$. Finally, suppose $g \in \mathcal{C}(\mathrm{I}_0)$. Then

$$\begin{aligned}(1 + \| H \|)^{q'} &|\psi_g(H; d_1 d_2)| \\ &\leqq (1 + \| H_0 \|)^{q'} (1 + \| H_- \|)^{q'} |\psi_{d_1 g}(H; d_2)| \\ &\leqq c^{q'} (1 + \| H_0 \|)^{q'} \sum_{1 \leqq j \leqq r} \nu_{N+q'}(d_j' (d_1 g)_{H_0}; \mathfrak{m}_0') \\ &\leqq 2^{N+q'} c^{q'} \sum_{1 \leqq j \leqq r} \tau_{N+q'}(d_j' d_1 g; \mathrm{I}_0)\end{aligned}$$

since

$$(1 + \| H_0 \|)^{q'} \tau_{N+q'}((d_j' d_1 g)_{H_0}; \mathfrak{m}_0') \leqq \tau_{N+2q'}(d_j' d_1 g; \mathrm{I}_0).$$

This shows that in this case $\psi_g \in \mathcal{C}(\mathfrak{h}_0'')$ and $g \to \psi_g$ $(g \in \mathcal{C}(\mathrm{I}_0))$ is a continuous mapping of $\mathcal{C}(\mathrm{I}_0)$ into $\mathcal{C}(\mathfrak{h}_0'')$. Thus we have proved the following result.

LEMMA 19. *For any $g \in \mathcal{S}(\mathrm{I}_0)$, let ψ_g denote the function on $\mathfrak{h}_0''$ defined by*

$$\psi_g(H) = \pi_{\mathfrak{m}}(H) \int_M g(mH)\, dm \qquad (H \in \mathfrak{h}_0'').$$

Then $\psi_g \in \mathcal{S}(\mathfrak{h}_0'')$ and $g \to \psi_g$ is a continuous mapping of $\mathcal{S}(\mathrm{I}_0)$ into $\mathcal{S}(\mathfrak{h}_0'')$. Moreover, $\psi_{\partial(p)g} = \partial(\tilde{p})\psi_g$ for $p \in I(\mathrm{I})$ and $g \in \mathcal{S}(\mathrm{I}_0)$. Finally, if $g \in \mathcal{C}(\mathrm{I}_0)$, then $\psi_g \in \mathcal{C}(\mathfrak{h}_0'')$ and the mapping $g \to \psi_g$ $(g \in \mathcal{C}(\mathrm{I}_0))$ of $\mathcal{C}(\mathrm{I}_0)$ into $\mathcal{C}(\mathfrak{h}_0'')$ is also continuous.

Now for $f \in \mathcal{S}(\mathfrak{g}_0)$ and $H \in \mathfrak{h}_0'$, put

$$\phi_f(H) = \pi(H) \int_{G^*} f(x^* H)\, dx^*.$$

That this integral is well defined is seen as follows. Put

$$f_1(X) = \int_K |f(kX)|\,dk \qquad (X \in \mathfrak{g}_0).$$

Then

$$\int_{G^*} |f(x^*H)|\,dx^* = |\zeta(H)|^{-1} \int_{M\times\mathfrak{n}_0} f_1(mH+Z)\,dm\,dZ$$

from the corollary to Lemma 15. Select an integer r such that

$$c = \int_{\mathfrak{n}_0} (1+\|Z\|)^{-r}\,dZ < \infty.$$

Then if $g(X) = \int_{\mathfrak{n}_0} f_1(X+Z)\,dZ \quad (X \in \mathfrak{l}_0)$,

$$(1+\|X+\theta(X)\|)^q g(X) \leqq c\nu_{q+r}(f;\mathfrak{g}_0)$$

for any integer $q \geqq 0$. Hence by applying Lemma 7 to $\mathfrak{m}_0'$, we conclude that $\int_M g(mH)\,dm < \infty$. This proves that the above integral for ϕ_f is convergent. For any $f \in \mathcal{S}(\mathfrak{g}_0)$, let $\bar{f}$ denote the function

$$\bar{f}(X) = \int_K f(kX)\,dk \qquad (X \in \mathfrak{g}_0).$$

Since $\theta(kX) = k\theta(X)$, it is easy to see that $\bar{f} \in \mathcal{S}(\mathfrak{g}_0)$ and $f \to \bar{f}$ is a continuous mapping of $\mathcal{S}(\mathfrak{g}_0)$ into itself. (Similarly if $f \in \mathcal{C}(\mathfrak{g}_0)$, one proves easily that $\bar{f} \in \mathcal{C}(\mathfrak{g}_0)$ and $f \to \bar{f}$ $(f \in \mathcal{C}(\mathfrak{g}_0))$ is a continuous mapping of $\mathcal{C}(\mathfrak{g}_0)$ into itself). Moreover, the above calculation shows that

$$\begin{aligned}\phi_f(H) &= \pi(H)\,|\pi_+(H)|^{-1} \int_{M\times\mathfrak{n}_0} \bar{f}(mH+Z)\,dm\,dZ \\ &= \epsilon_+(H)\pi_{\mathfrak{m}}(H) \int_{M\times\mathfrak{n}_0} \bar{f}(mH+Z)\,dm\,dZ \qquad (f \in \mathcal{S}(\mathfrak{g}_0)),\end{aligned}$$

where $\epsilon_+(H) = 1$ or -1 according as $\pi_+(H)$ is positive[10] or negative. Put

$$\psi_f(H) = \pi_{\mathfrak{m}}(H) \int_{M\times\mathfrak{n}_0} \bar{f}(mH+Z)\,dm\,dZ \qquad (H \in \mathfrak{h}_0'', f \in \mathcal{S}(\mathfrak{g}_0)).$$

It follows from Lemmas 17 and 19 that $\psi_f \in \mathcal{S}(\mathfrak{h}_0'')$ and $f \to \psi_f$ is a continuous mapping of $\mathcal{S}(\mathfrak{g}_0)$ into $\mathcal{S}(\mathfrak{h}_0'')$. Define g_f as in Lemma 17. Then

$$\psi_f(H) = \pi_{\mathfrak{m}}(H) \int_M g_f(mH)\,dm \qquad (f \in \mathcal{S}(\mathfrak{g}_0), H \in \mathfrak{h}_0'').$$

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[10] Notice that $\pi_+(H)$ is real since it is equal to the determinant of the restriction of ad H on $\mathfrak{n}_0$

Now suppose $p \in I(\mathfrak{g})$. If $f_1 = \partial(p)f$, it is obvious that $\tilde{f}_1 = \partial(p)\tilde{f}$. Hence

$$\psi_{\partial(p)f}(H) = \pi_m(H) \int_M g_{\tilde{f}}(mH; \partial(p_I))\, dm = \psi_f(H; \partial(\tilde{p}))$$

from Lemmas 17 and 19. Finally, if $f \in \mathcal{B}(\mathfrak{g}_0)$, we conclude from these two lemmas that $\psi_f \in \mathcal{B}(\mathfrak{h}_0'')$ and the mapping $f \to \psi_f$ $(f \in \mathcal{B}(\mathfrak{g}_0))$ of $\mathcal{B}(\mathfrak{g}_0)$ into $\mathcal{B}(\mathfrak{h}_0'')$ is continuous. Thus we have obtained the following theorem.

THEOREM 3. *For any $f \in \mathcal{S}(\mathfrak{g}_0)$, the integral*

$$\phi_f(H) = \pi(H) \int_{G^*} f(x^*H)\, dx^*$$

is absolutely convergent for $H \in \mathfrak{h}_0'$ and the function ϕ_f, so defined, lies in $\mathcal{S}(\mathfrak{h}_0')$. Moreover, $f \to \phi_f$ is a continuous mapping of $\mathcal{S}(\mathfrak{g}_0)$ into $\mathcal{S}(\mathfrak{h}_0')$, and

$$\phi_{\partial(p)f} = \partial(\tilde{p})\phi_f$$

for $p \in I(\mathfrak{g})$. Let $\mathfrak{h}_0''$ be the set of those points $H \in \mathfrak{h}_0$ where $\prod_{\alpha \in P_0} \alpha(H) \neq 0$ and let $\epsilon_+(H)$ denote the sign of $\pi_+(H)$ $(H \in \mathfrak{h}_0')$. Then $\epsilon_+\phi_f$ $(f \in \mathcal{S}(\mathfrak{g}))$ can be extended to a function of class C^∞ on $\mathfrak{h}_0''$. Finally, $\phi_f \in \mathcal{B}(\mathfrak{h}_0')$ if $f \in \mathcal{B}(\mathfrak{g}_0)$, and the mapping $f \to \phi_f$ of $\mathcal{B}(\mathfrak{g}_0)$ into $\mathcal{B}(\mathfrak{h}_0')$ is also continuous.

Define the homomorphism δ of $\mathfrak{Z}(\mathfrak{g})$ into $\mathfrak{Z}(\mathfrak{h})$ as in Theorem 1 of [2(h)].

COROLLARY. *If $\xi \in \mathfrak{Z}(\mathfrak{g})$ and $f \in \mathcal{B}(\mathfrak{g}_0)$, then $\phi_{\xi f} = \delta(\xi)\phi_f$.*

From the above theorem, this is true if $\xi \in \partial(I(\mathfrak{g}))$. Since $\mathfrak{Z}(\mathfrak{g})$ is generated by $I(\mathfrak{g}) \cup \partial(I(\mathfrak{g}))$, it is enough to verify the above statement for $\xi \in I(\mathfrak{g})$. But then $\xi f \in \mathcal{B}(\mathfrak{g}_0)$, and it is obvious from the definition of ϕ_f that

$$\phi_{\xi f} = \tilde{\xi}\phi_f = \delta(\xi)\phi_f.$$

6. Fourier transforms. Let dX denote the regular Euclidean measure on $\mathfrak{g}_0$ (see [2(h), §2]) and for any $f \in \mathcal{B}(\mathfrak{g}_0)$, let $\tilde{f}$ denote the Fourier transform of f so that

$$\tilde{f}(Y) = \int_{\mathfrak{g}_0} \exp\{(-1)^{\frac{1}{2}} B(Y, X)\} f(X)\, dX \qquad (Y \in \mathfrak{g}_0).$$

Fix an element $H_0 \in \mathfrak{h}_0'$ and consider the mapping $T_{H_0}: f \to \phi_{\tilde{f}}(H_0)$ $(f \in \mathcal{B}(\mathfrak{g}_0))$. It follows from Theorem 3 that T_{H_0} is a $\mathcal{B}$-distribution. We intend to study this distribution more closely. For any $p \in S(\mathfrak{g})$, let $\hat{p}$ denote the

polynomial given by $\hat{p}(X) = p(-(-1)^{\frac{1}{2}}X)$ $(X \in \mathfrak{g})$. Then (see Lemma 1 of [2(h)]) $(\partial(p)f)^{\sim} = \hat{p}\tilde{f}$ and $(pf)^{\sim} = \partial(\hat{p})\tilde{f}$. Moreover, if $p \in I(\mathfrak{g})$, the same holds for $\hat{p}$. Therefore it follows from Theorem 3 and its corollary that

$$T_{H_0}(\partial(p)f) = \hat{p}(H_0)T_{H_0}(f) \qquad (f \in \mathcal{C}(\mathfrak{g}_0), p \in I(\mathfrak{g})),$$

and this proves that $\partial(p)T_{H_0} = p((-1)^{\frac{1}{2}}H_0)T_{H_0}$ for $p \in I(\mathfrak{g})$. Thus we have obtained the following lemma.

LEMMA 20. *For any $p \in S(\mathfrak{g})$, define the polynomial p' by $p'(X) = p((-1)^{\frac{1}{2}}X)$ $(X \in \mathfrak{g})$. Then $\partial(p)T_{H_0} = p'(H_0)T_{H_0}$ for $p \in I(\mathfrak{g})$.*

Let l be the rank and n the dimension of $\mathfrak{g}$. Then if λ is an indeterminate and I the identity mapping of $\mathfrak{g}$,

$$\det(\lambda I - \operatorname{ad} X) = \lambda^n + d_1(X)\lambda^{n-1} + d_2(X)\lambda^{n-2} + \cdots + d_{n-l}(X)\lambda^l \quad (X \in \mathfrak{g}),$$

where $d_j \in I(\mathfrak{g})$ and $d_{n-l} \neq 0$. It is obvious that the polynomials d_j take only real values on $\mathfrak{g}_0$ and $d_{n-l}(H) = (-1)^r\pi(H)^2$ $(H \in \mathfrak{h})$, where $r = \frac{1}{2}(n-l)$ is the number of positive roots. Put $\eta = (-1)^r d_{n-l}$. Then (see Lemma 9 of [2(h)]) η is the unique element in $I(\mathfrak{g})$ which coincides with π^2 on $\mathfrak{h}$. Moreover, as we have already observed above, η takes only real values on $\mathfrak{g}_0$. *We call a point $X \in \mathfrak{g}$ singular or regular according as $\eta(X)$ is zero or not.* Let $\mathfrak{g}_0'$ be the set of all regular elements in $\mathfrak{g}_0$. Then $\mathfrak{g}_0'$ is an open dense subset of $\mathfrak{g}_0$. Our object is to show that T_{H_0} coincides with an analytic function on $\mathfrak{g}_0'$.

Let $\bar{H}_0$ be a regular element in $\mathfrak{g}_0$ and $\bar{\mathfrak{h}}_0$ the centralizer of $\bar{H}_0$ in $\mathfrak{g}_0$. Then $\bar{\mathfrak{h}}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ (see Weyl [8]). Let $\bar{A}$ be the Cartan subgroup of G corresponding to $\bar{\mathfrak{h}}_0$. Put $\bar{G} = G/\bar{A}$ and let $x \to \bar{x}$ denote the natural mapping of G on $\bar{G}$. Complexify $\bar{\mathfrak{h}}_0$ to $\bar{\mathfrak{h}}$, and let $\bar{P}$ be the set of all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ under some fixed order. Put $\bar{\pi} = \prod_{\alpha \in \bar{P}} \alpha$, $\bar{\mathfrak{h}}_0' = \bar{\mathfrak{h}}_0 \cap \mathfrak{g}_0'$ and $\bar{x}\bar{H} = x\bar{H}$ $(x \in G, \bar{H} \in \bar{\mathfrak{h}})$. It is well known (see [2(d), p. 501]) that the mapping $\psi: (\bar{x}, \bar{H}) \to \bar{x}\bar{H}$ of $\bar{G} \times \bar{\mathfrak{h}}_0$ into $\mathfrak{g}_0$ is everywhere regular on $\bar{G} \times \bar{\mathfrak{h}}_0'$. Let $\bar{A}$ be the normalizer of $\bar{\mathfrak{h}}_0$ in G. Then $\bar{A}/\bar{A}$ can be regarded (see [2(e), §2]) as a subgroup of the Weyl group $\bar{W}$ of $\mathfrak{g}$ with respect $\bar{\mathfrak{h}}$, and therefore it is finite. Since $\bar{A}$ is the centralizer of $\bar{H}_0$ in G (Lemma 3 of [2(e)]), it is clear that we can select an open connected neighborhood V of $\bar{H}_0$ in $\bar{\mathfrak{h}}_0'$ such that ψ is univalent on $\bar{G} \times V$. Put $\mathfrak{g}_1 = \psi(\bar{G} \times V)$. Then $\mathfrak{g}_1$ is an open connected neighborhood of $\bar{H}_0$ in $\mathfrak{g}_0'$ and ψ defines an analytic isomorphism of $\bar{G} \times V$ with $\mathfrak{g}_1$. Let $d\bar{x}$ denote the invariant measure on $\bar{G}$ and $d\bar{H}$ the regular Euclidean

measure on $\bar{\mathfrak{h}}_0$. Then a simple calculation shows (see [2(d), p. 501]) that $d\bar{x}$ can be so normalized that

$$dX = |\eta(\bar{H})|\, d\bar{x}\, d\bar{H} = |\bar{\pi}(\bar{H})|^2\, d\bar{x}\, d\bar{H} \qquad (X = \bar{x}\bar{H})$$

for $(\bar{x}, \bar{H}) \in \bar{G} \times V$.

Now let T be a distribution on $\mathfrak{g}_0'$ which is invariant under G (see [2(h), §2]). For any $g \in C_c^\infty(\bar{G} \times V)$, let F_g denote the function in $C_c^\infty(\mathfrak{g}_1)$ given by $F_g(\bar{x}\bar{H}) = g(\bar{x}, \bar{H})$ $(\bar{x} \in \bar{G}, \bar{H} \in V)$. Then we can define a distribution τ on $\bar{G} \times V$ by $\tau(g) = T(F_g)$ $(g \in C_c^\infty(\bar{G} \times V))$. For $\alpha \in C_c^\infty(\bar{G})$ and $\beta \in C_c^\infty(V)$, let $\alpha \times \beta$ denote the function g in $C_c^\infty(\bar{G} \times V)$ given by $g(\bar{x}, \bar{H}) = \alpha(\bar{x})\beta(\bar{H})$ $(\bar{x} \in \bar{G}, \bar{H} \in V)$.

LEMMA 21. *There exists a distribution S on V such that $\tau(\alpha \times \beta) = \int \alpha(\bar{x})\, d\bar{x}\, S(\beta)$ for all $\alpha \in C_c^\infty(\bar{G})$ and $\beta \in C_c^\infty(V)$.*

Let dx and da denote the Haar measures of G and $\bar{A}$, respectively. We assume that da is so normalized that

$$\int g(x)\, dx = \int_{\bar{G}} d\bar{x} \int_{\bar{A}} g(xa)\, da \qquad (g \in C_c(G)).$$

For any $f \in C_c^\infty(G \times V)$, let $\bar{f}$ denote the function in $C_c^\infty(\bar{G} \times V)$ given by

$$\bar{f}(\bar{x}, \bar{H}) = \int_{\bar{A}} f(xa, \bar{H})\, da.$$

Then we can define a distribution τ' on $G \times V$ by $\tau'(f) = \tau(\bar{f})$ $(f \in C_c^\infty(G \times V))$. For any $\alpha \in C_c^\infty(G)$, define $\alpha' \in C_c^\infty(\bar{G})$ by

$$\alpha'(\bar{x}) = \int_{\bar{A}} \alpha(xa)\, da \qquad (x \in G).$$

Then one sees without difficulty that $\alpha \to \alpha'$ is a continuous mapping of $C_c^\infty(G)$ onto $C_c^\infty(\bar{G})$. Hence if we denote the function $(x, \bar{H}) \to \alpha(x)\beta(\bar{H})$ on $G \times V$ by $\alpha \times \beta$ $(\alpha \in C_c^\infty(G),\ \beta \in C_c^\infty(V))$, it is enough to prove the existence of a distribution S on V such that

$$\tau'(\alpha \times \beta) = \int \alpha(x)\, dx\, S(\beta) \qquad (\alpha \in C_c^\infty(G),\ \beta \in C_c^\infty(V)).$$

Now fix β. Then $\sigma_\beta : \alpha \to \tau'(\alpha \times \beta)$ is a distribution on G. Moreover, since T is invariant under G, it is clear that σ_β is invariant under left translations of G. Therefore it follows from Lemma 36 of [2(e)], that

$$\tau'(\alpha \times \beta) = S(\beta) \int_G \alpha(x)\, dx,$$

where $S(\beta)$ is a complex number independent of α. Now select $\alpha_0 \in C_c^\infty(G)$ such that $\int_G \alpha_0(x)dx = 1$. Then since $\beta \to \tau'(\alpha_0 \times \beta) = S(\beta)$ $(\beta \in C_c^\infty(V))$ is obviously a distribution on V, our assertion follows.

Since $\eta(\bar{H}) = \bar{\pi}(\bar{H})^2$ $(\bar{H} \in \bar{\mathfrak{h}})$, $\bar{\pi}$ is nowhere zero on V. Hence $\sigma = (\bar{\pi})^{-1}S$ is also a distribution on V.

LEMMA 22. *Let p be a homogeneous element in $I(\mathfrak{g})$ and let $\tilde{p}$ denote its restriction on $\mathfrak{h}$. Suppose $\partial(p)T = cT$ $(c \in C)$. Then $\partial(\tilde{p})\sigma = c\sigma$.*

Select $\alpha_0 \in C_c^\infty(G)$ such that $\int \alpha_0(x)dx = 1$, and, for any $\beta \in C_c^\infty(V)$, let F_β denote the function in $C_c^\infty(\mathfrak{g}_1)$ given by

$$F_\beta(\bar{x}\bar{H}) = \beta(\bar{H})\alpha_0'(\bar{x}) \qquad (\bar{x} \in \bar{G}, \bar{H} \in V),$$

where $\alpha_0'(\bar{x}) = \int_{\bar{A}} \alpha_0(xa)da$ $(x \in G)$. We put $F_\beta(x : \bar{H}) = F_\beta(x\bar{H})$ $(x \in G, \bar{H} \in V)$ and use the notation of Lemma 3 of [2(h)]. Then it follows from Lemmas 5 and 6 of [2(h)], that

$$F_\beta(x\bar{H}\,;\partial(p)) = \bar{\pi}(\bar{H})^{-1}F_\beta\,(x : \bar{H}\,;\partial(\tilde{p}) \circ \bar{\pi}) + \sum_{1 \leqq i \leqq N} a_i(\bar{H})F_\beta(x\,; b_i : \bar{H}\,;\partial(\bar{u}_i))$$

$(x \in G, \bar{H} \in V)$, where $b_i \in \mathfrak{Bg}$, $\bar{u}_i \in S(\bar{\mathfrak{h}})$ and the a_i are analytic functions on V $(1 \leqq i \leqq N)$. Moreover, in view of Lemma 22 of [2(h)], we can assume that $b_i^a = b_i$ $(a \in \bar{A})$. On the other hand,

$$F_\beta(x : \bar{H}) = \int \alpha_0(xa)da\,\beta(\bar{H}).$$

Therefore

$$F_\beta(\bar{x}\bar{H}\,;\partial(p)) = \alpha_0'(\bar{x})\bar{\pi}(\bar{H})^{-1}\beta(\bar{H}\,;\partial(\tilde{p}) \circ \bar{\pi}) + \sum_{1 \leqq i \leqq N} a_i(\bar{H}) \int \alpha_0(xa\,; b_i)da\;\beta(\bar{H}\,;\partial(\bar{u}_i)).$$

But

$$\int d\bar{x} \int \alpha_0(xa\,; b_i)da = \int \alpha_0(x\,; b_i)dx = 0$$

since $b_i \in \mathfrak{Bg}$. Therefore

$$T(\partial(p)F_\beta) = \sigma(\partial(\tilde{p})\,(\bar{\pi}\beta)).$$

Let m be the degree of p. Then

$$T(\partial(p)F_\beta) = (-1)^m cT(F_\beta) = (-1)^m c\sigma(\bar{\pi}\beta),$$

and this shows that

$$\sigma(\partial(\bar{p})(\overline{\pi}\beta)) = (-1)^m c\sigma(\overline{\pi}\beta).$$

But since $\overline{\pi}$ is nowhere zero on V, this implies that $\partial(\bar{p})\sigma = c\sigma$.

COROLLARY. *Suppose T is an eigen-distribution of $\partial(p)$ for every $p \in I(\mathfrak{g})$. Then σ coincides with an analytic function on V.*

This follows from Lemma 12 of [2(h)] and the above result.

We shall now apply this corollary to the distribution $T = T_{H_0}$ of Lemma 20. Let G_c be the (connected) complex adjoint group of $\mathfrak{g}$. Since any two Cartan subalgebras of $\mathfrak{g}$ are conjugate under G_c (see [2(e), Lemma 4]), we can select $y \in G_c$ such that $y\mathfrak{h} = \bar{\mathfrak{h}}$. Then if $p \in I(\mathfrak{g})$, $p^y = p$, and therefore $p'(H_0) = p((-1)^{\frac{1}{2}} yH_0)$. In particular, $\eta(yH_0) = \eta(H_0) \neq 0$, and so $\overline{\pi}((-1)^{\frac{1}{2}} yH_0) \neq 0$. Therefore it follows from Lemma 20 and the Corollary of Lemma 13 of [2(h)], that

$$\sigma(\bar{H}) = \sum_{s \in \overline{W}} c_s \exp\{(-1)^{\frac{1}{2}} B(s\bar{H}, yH_0)\} \qquad (\bar{H} \in V),$$

where the c_s are certain complex numbers. Moreover, we can conclude from Lemma 21 (see Schwartz [6(I), p. 108]) that

$$T_{H_0}(f) = \int_{\bar{G} \times V} f(\bar{x}\bar{H})\, \overline{\pi}(\bar{H})\, \sigma(\bar{H})\, d\bar{x}\, d\bar{H} \qquad (f \in C_c^\infty(\mathfrak{g}_1)).$$

Now put

$$\bar{\phi}_f(\bar{H}) = \overline{\pi}(\bar{H}) \int_{\bar{G}} f(\bar{x}\bar{H})\, d\bar{x} \qquad (\bar{H} \in \bar{\mathfrak{h}}_0').$$

Then the above relation becomes

$$\phi\tilde{f}(H_0) = \sum_{s \in \overline{W}} c_s(H_0) \int \bar{\phi}_f(\bar{H}) \exp\{(-1)^{\frac{1}{2}} B(s\bar{H}, yH_0)\}\, d\bar{H} \qquad (H_0 \in \mathfrak{h}_0')$$

for $f \in C_c^\infty(\mathfrak{g}_1)$. (Since the numbers c_s may depend on H_0, we have now written them as $c_s(H_0)$.) Let $\mathfrak{h}_1$ be a connected component of $\mathfrak{h}_0'$. We shall prove that the c_s are actually constant on $\mathfrak{h}_1$. Let H_0 be a fixed point in $\mathfrak{h}_1$ and U a sufficiently small open connected neighborhood of H_0 in $\mathfrak{h}_1$. It is enough to prove that the c_s are constant on U. Since $\eta(yH_0) = \eta(H_0) \neq 0$, it is clear that $syH_0 \neq yH_0$ for any $s \neq 1$ in $\bar{W}$ (see Lemma 4 of [2(e)]). Hence we can choose polynomials $q_s \in S(\bar{\mathfrak{h}})$ $(s \in \bar{W})$ such that

$$q_s(-(-1)^{\frac{1}{2}} t^{-1} yH_0) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases} \qquad (t \in \bar{W}).$$

Furthermore, select $\beta_s \in C_c^\infty(V)$ such that

$$\int_V \exp\{(-1)^{\frac{1}{2}} B(yH_0, s\bar{H})\}\beta_s(\bar{H})\,d\bar{H} = 1 \qquad (s \in \bar{W}).$$

Fix an element $\alpha \in C_c^\infty(\bar{G})$ such that $\int \alpha(\bar{x})\,d\bar{x} = 1$, and define $f_s \in C_c^\infty(\mathfrak{g}_1)$ by $f_s(\bar{x}\bar{H}) = \alpha(\bar{x})\beta_s(\bar{H}; \partial(q_s))\bar{\pi}(\bar{H})^{-1}$ $(\bar{x} \in \bar{G}, \bar{H} \in V)$. Then if $H \in U$,

$$\begin{aligned}\phi_{\tilde{f}_s}(H) &= \sum_{t \in \bar{W}} c_t(H) \int \beta_s(\bar{H}; \partial(q_s))\exp\{(-1)^{\frac{1}{2}} B(t\bar{H}, yH)\}\,d\bar{H} \\ &= \sum_{t \in \bar{W}} c_t(H)\, a_{s,t}(H) \qquad (s \in \bar{W}),\end{aligned}$$

where

$$a_{s,t}(H) = q_s(-(-1)^{\frac{1}{2}} t^{-1} yH) \int \beta_s(\bar{H})\exp\{(-1)^{\frac{1}{2}} B(t\bar{H}, yH)\}\,d\bar{H}.$$

But it is clear that $a_{s,t}$ are analytic functions on U and

$$a_{s,t}(H_0) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

Therefore if U is sufficiently small, the determinant of the matrix $(a_{s,t})_{s,t \in \bar{W}}$ is nowhere zero on U. Let $(b_{s,t})_{s,t \in \bar{W}}$ denote the inverse matrix. Then the $b_{s,t}$ are also analytic functions on U and

$$c_t(H) = \sum_{s \in \bar{W}} b_{t,s}(H)\phi_{\tilde{f}_s}(H) \qquad (H \in U)$$

Therefore since $\phi_{\tilde{f}_s}$ are of class C^∞ on U (Theorem 3), the same holds for c_t.

Put

$$\sigma(H : \bar{H}) = \sum_{s \in \bar{W}} c_s(H)\exp\{(-1)^{\frac{1}{2}} B(yH, s\bar{H})\} \qquad (H \in U, \bar{H} \in V).$$

Then σ is of class C^∞ on $U \times V$. On the other hand, $\partial(p)\tilde{f} = (p'f)^{\sim}$ $(p \in S(\mathfrak{g}), f \in C_c^\infty(\mathfrak{g}))$, and therefore if $p \in I(\mathfrak{g})$,

$$\phi_{(p'f)^{\sim}} = \partial(\bar{p})\phi_{\tilde{f}}$$

from Theorem 3. Select α as above and define $f \in C_c^\infty(\mathfrak{g}_1)$ by $f(\bar{x}\bar{H}) = \alpha(\bar{x})\beta(\bar{H})\bar{\pi}(\bar{H})^{-1}$ $(\beta \in C_c^\infty(V))$. Then $\bar{\phi}_f(\bar{H}) = \beta(\bar{H})$ and $\bar{\phi}_{p'f}(\bar{H}) = p'(\bar{H})\beta(\bar{H})$ $(\bar{H} \in V)$, and so the above equation becomes

$$\int p'(\bar{H})\beta(\bar{H})\sigma(H : \bar{H})\,d\bar{H} = \int \beta(\bar{H})\sigma(H; \partial(\bar{p}) : \bar{H})\,d\bar{H} \qquad (H \in U).$$

As this is true for every $\beta \in C_c^\infty(V)$, we conclude that

$$\sigma(H; \partial(\bar{p}) : \bar{H}) = p'(\bar{H})\sigma(H : \bar{H}) \qquad (p \in I(\mathfrak{g}))$$

for $H \in U$ and $\bar{H} \in V$. Thus it is enough to prove the following result.

LEMMA 23. *Suppose* c_s $(s \in \bar{W})$ *are functions of class* C^∞ *on* U *and*

$$\sigma(H:\bar{H}) = \sum_{s\in\bar{W}} c_s(H)\exp\{(-1)^{\frac{1}{2}}B(yH, s\bar{H})\} \qquad (H\in U, \bar{H}\in V).$$

Then if $\sigma(H;\partial(\bar{p}):\bar{H}) = p'(\bar{H})\sigma(H:\bar{H})$ *for all* $p\in I(\mathfrak{g})$ *and* $(H,\bar{H})\in U\times V$, *it follows that each* c_s *is constant on* U.

Fix $\bar{H}$ in V for a moment, and let μ_s denote the linear function $H\to(-1)^{\frac{1}{2}}B(s\bar{H}, yH)$ $(H\in\mathfrak{h})$ on $\mathfrak{h}$. Since $\bar{\pi}(\bar{H})\neq 0$, the functions μ_s $(s\in W)$ are all distinct. Therefore e^{μ_s} are linearly independent over $S(\mathfrak{h})$ (Lemma 41 of [2(b)]). This means that the coefficients of e^{μ_s} in $\sigma(H;\partial(\bar{p}):\bar{H}) - p'(\bar{H})\sigma(H:\bar{H})$ must all be zero. But it is obvious that for a fixed s, this coefficient is of the form

$$c_s(H;\partial(\bar{p})) + \sum_{1\leqq i\leqq N} q_i(\bar{H})b_i(H) - p'(\bar{H})c_s(H),$$

where the q_i are certain homogeneous elements in $S(\bar{\mathfrak{h}})$ of positive degree and the b_i are functions of class C^∞ on U. Since $\bar{H}$ was an arbitrary element of V, we conclude that

$$c_s(H;\partial(\bar{p})) = 0 \qquad (s\in\bar{W}, H\in U)$$

for all homogeneous elements $p\in I(\mathfrak{g})$ of positive degree. But then it follows from Lemmas 12 and 14 of [2(h)] that c_s is a polynomial function. Now again fix $\bar{H}\in V$. Then, in view of the equation $\sigma(H;\partial(\bar{p}):\bar{H}) = p'(\bar{H})\sigma(H:\bar{H})$ $(p\in I(\mathfrak{g}))$, we conclude from the Corollary of Lemma 13 of [2(h)] that

$$\sigma(H:\bar{H}) = \sum_{s\in W} b_s e^{\mu_s(H)} \qquad (H\in U),$$

where the b_s are certain complex numbers independent of H. However the e^{μ_s} are linearly independent over $S(\mathfrak{h})$, and therefore $c_s = b_s$ $(s\in\bar{W})$. This proves that the c_s are constant on U.

Thus we have obtained the following result which should be compared with Theorem 3 of [2(h)].

LEMMA 24. *Let* $\mathfrak{h}_1$ *be a connected component of* $\mathfrak{h}_0'$. *Then there exist complex numbers* c_s $(s\in\bar{W})$ *such that*

$$\phi\tilde{f}(H) = \sum_{s\in\bar{W}} c_s \int \bar{\phi}_f(\bar{H})\exp\{(-1)^{\frac{1}{2}}B(s\bar{H}, yH)\}d\bar{H} \qquad (H\in\mathfrak{h}_1)$$

for all $f\in C_c^\infty(\mathfrak{g}_1)$.

For any $X\in\mathfrak{g}_0'$, put $\epsilon(X) = 1$ or -1 according as $\eta(X)$ is positive or

negative. Then it is clear that for a fixed $H \in \mathfrak{h}_1$, there exists an analytic function F_H on $\mathfrak{g}_1$ such that

$$F_H(\bar{x}\bar{H}) = \epsilon(\bar{H})\bar{\pi}(\bar{H})^{-1} \sum_{s \in \bar{W}} c_s \exp\{(-1)^{\frac{1}{2}} B(s\bar{H}, yH)\}$$

$(\bar{x} \in \bar{G}, \bar{H} \in V)$ and

$$\phi_{\tilde{f}}(H) = \int f(X) F_H(X)\, dX \qquad (f \in C_c^\infty(\mathfrak{g}_1)).$$

(Compare this formula for F_H with Theorem 2 of [2(h)]). This shows that the distribution $f \to \phi_{\tilde{f}}(H)$ coincides on $\mathfrak{g}_1$ with F_H. Thus we have proved the following theorem.

THEOREM 4. *For a fixed $H \in \mathfrak{h}_0'$ consider the $\mathcal{B}$-distribution $T_H: f \to \phi_{\tilde{f}}(H)$ $(f \in \mathcal{B}(\mathfrak{g}_0))$ on $\mathfrak{g}_0$. Then T_H coincides on $\mathfrak{g}_0'$ with an analytic function F_H.*

Since T_H is invariant under G, the same holds for F_H. The exact form of F_H has already been obtained above.

7. The distributions T and T'. Let $H_1, \cdots, H_l$ be a base for $\mathfrak{h}_0$ over R. It is clear that there exists a positive constant c such that if $H = t_1 H_1 + \cdots + t_l H_l$ $(t_i \in R)$, then $|t_i| \leqq c \| H \|$ $(1 \leqq i \leqq l)$. Now suppose $\mathfrak{h}_1$ is a connected component of $\mathfrak{h}_0'$ and $f \in \mathcal{B}(\mathfrak{h}_1)$. Then $\partial(H)f = \sum_{1 \leqq i \leqq l} t_i \partial(H_i) f$ and therefore

$$\nu(\partial(H)f) \leqq c \| H \| \sum_{1 \leqq i \leqq l} \nu(\partial(H_i)f) \qquad (H \in \mathfrak{h}_0),$$

where $\nu(g) = \sup_{H \in \mathfrak{h}_1} |g(H)|$ $(g \in \mathcal{B}(\mathfrak{h}_1))$. Suppose H, H' are two points in $\mathfrak{h}_1$ and the straight line-segment joining them lies entirely in $\mathfrak{h}_1$. Then if $f \in \mathcal{B}(\mathfrak{h}_1)$,

$$f(H') - f(H) = \int_0^1 (d/dt) f(H + t(H' - H))\, dt,$$

and therefore

$$|f(H') - f(H)| \leqq \nu(\partial(H' - H)f) \leqq c \| H' - H \| \sum_{1 \leqq i \leqq l} \nu(\partial(H_i)f).$$

But in view of Lemma 43 of the Appendix (§12), we can conclude that these inequalities are actually valid for any two points $H, H' \in \mathfrak{h}_1$, and so we have the following result.

LEMMA 25. *Let $\mathrm{Cl}(\mathfrak{h}_1)$ denote the closure of $\mathfrak{h}_1$ in $\mathfrak{h}_0$. Then every $f \in \mathcal{B}(\mathfrak{h}_1)$ can be extended to a continuous function on $\mathrm{Cl}(\mathfrak{h}_1)$.*

For any $\phi \in \mathcal{B}(\mathfrak{h}_0')$ and $\xi \in \mathcal{D}(\mathfrak{h})$, put

$$\lim_{\mathfrak{h}_1} \phi(0) = \lim_{H \to 0} \phi(H), \qquad \lim_{\mathfrak{h}_1} \phi(0;\xi) = \lim_{H \to 0} \phi(H;\xi) \qquad (H \in \mathfrak{h}_1).$$

It is clear from the above corollary that these limits exist and

$$|\lim_{\mathfrak{h}_1} \phi(0;\xi)| \leqq \nu(\xi\phi).$$

Therefore for a fixed ξ, the mapping $\phi \to \lim_{\mathfrak{h}_1} \phi(0;\xi)$ is continuous on $\mathcal{B}(\mathfrak{h}_0')$. Let ξ_0 denote the local expression of ξ at zero. Then if $\xi_0 = 0$, it is obvious that $\lim_{\mathfrak{h}_1} \phi(0;\xi) = 0$. Hence

$$\lim_{\mathfrak{h}_1} \phi(0;\xi) = \lim_{\mathfrak{h}_1} \phi(0;\xi_0)$$

for $\phi \in \mathcal{B}(\mathfrak{h}_0')$ and $\xi \in \mathcal{D}(\mathfrak{h})$.

We know from Theorem 3 that if $f \in \mathcal{B}(\mathfrak{g}_0)$, then $\phi_f \in \mathcal{B}(\mathfrak{h}_0')$. Hence we can consider the mapping $T: f \to \lim_{\mathfrak{h}_1} \phi_f(0;\partial(\pi))$ $(f \in \mathcal{B}(\mathfrak{g}_0))$. It follows immediately from Theorem 3 and what we have said above, that T is a $\mathcal{B}$-distribution.

LEMMA 26. *Define the polynomial $\eta \in I(\mathfrak{g})$ as in Section 6. Then*

$$\eta^k \partial(\eta^k) T = \langle \eta^k, \eta^k \rangle T$$

for any integer $k \geqq 0$.

Put $\xi = \partial(\eta^k) \circ \eta^k$. Then we know from the Corollary of Theorem 3 that $\phi_{\xi f} = \delta(\xi)\phi_f$ $(f \in \mathcal{B}(\mathfrak{g}_0))$, and therefore

$$T(\xi f) = \lim_{\mathfrak{h}_1} \phi_f(0;\partial(\pi) \circ \delta(\xi)) = \langle \eta^k, \eta^k \rangle T(f)$$

from the Corollary of Lemma 18 of [2(h)]. Our assertion now follows from the fact that the degree of η is even.

It is one of our principal objects to prove that $T(f) = cf(0)$ $(f \in \mathcal{B}(\mathfrak{g}_0)$, where c is a complex number independent of f. However, the proof of this fact is rather long and difficult, and we shall not be able to complete it in this paper. We consider first the Fourier transform T' of T given by $T'(f) = T(\check{f})$ $(f \in \mathcal{B}(\mathfrak{g}_0))$. Then T' is also a $\mathcal{B}$-distribution on $\mathfrak{g}_0$, and the above-mentioned fact about T is obviously equivalent to the statement that T' coincides on $\mathfrak{g}_0$ with a constant. This we shall first prove in a weaker form. As before, let $\mathfrak{g}_0'$ denote the set of regular elements in $\mathfrak{g}_0$.

LEMMA 27. *Let T' be the $\mathcal{B}$-distribution on $\mathfrak{g}_0$ defined by*

$$T'(f) = \lim_{\mathfrak{h}_1} \phi_{\check{f}}(0;\partial(\pi)) \qquad (f \in \mathcal{B}(\mathfrak{g}_0)).$$

Then on each connected component of $\mathfrak{g}_0'$, T' coincides with a constant.

It is obviously sufficient to prove that if $\bar{H}_0 \in \mathfrak{g}_0'$, then T' coincides on some open neighborhood of $\bar{H}_0$ in $\mathfrak{g}_0'$ with a constant. We use the notation of the proof of Lemma 24. Then

$$\phi_f^{\sim}(H) = \sum_{s \in \bar{W}} c_s \int_V \bar{\phi}_f(\bar{H}) \exp\{(-1)^{\frac{1}{2}} B(s\bar{H}, yH)\} d\bar{H} \qquad (H \in \mathfrak{h}_1)$$

for $f \in C_c^\infty(\mathfrak{g}_1)$. Hence

$$\phi_f^{\sim}(H; \vartheta(\pi)) = \sum_{s \in \bar{W}} c_s (-1)^{r/2} \int_V \bar{\phi}_f(\bar{H}) \pi(y^{-1}s\bar{H}) \exp\{(-1)^{\frac{1}{2}} B(s\bar{H}, yH)\} d\bar{H}$$

where r is the degree of π. This implies that

$$T'(f) = \sum_{s \in \bar{W}} c_s (-1)^{r/2} \int_V \bar{\phi}_f(\bar{H}) \pi(y^{-1}s\bar{H}) d\bar{H}.$$

But $\pi(y^{-1}s\bar{H})^2 = \eta(y^{-1}s\bar{H}) = \eta(\bar{H}) = \bar{\pi}(\bar{H})^2$. Therefore it is obvious that $\epsilon_s = \pi(y^{-1}s\bar{H})/\bar{\pi}(\bar{H})$ is constant for $\bar{H} \in V$. Hence

$$T'(f) = c' \int_V \bar{\pi}(\bar{H}) \bar{\phi}_f(\bar{H}) d\bar{H},$$

where $c' = \sum_{s \in \bar{W}} (-1)^{r/2} c_s \epsilon_s$. Now put $\epsilon = 1$ or -1 according as η is positive or negative on $\mathfrak{g}_1$. Then

$$\int_V \bar{\pi}(\bar{H}) \bar{\phi}_f(\bar{H}) d\bar{H} = \epsilon \int f(X) dX \qquad (f \in C_c^\infty(\mathfrak{g}_1)),$$

and therefore

$$T'(f) = c \int f(X) dX \qquad (f \in C_c^\infty(\mathfrak{g}_1))$$

if $c = c'\epsilon$. This proves that T' coincides with c on $\mathfrak{g}_1$.

LEMMA 28. *$\mathfrak{g}_0'$ has only a finite number of connected components.*

From the corollary of Lemma 2 of [2(e)], we can select a finite number of Cartan subalgebras $\mathfrak{h}_{(i)}$ $(1 \leqq i \leqq N)$ such that every Cartan subalgebra of $\mathfrak{g}_0$ is conjugate to some $\mathfrak{h}_{(i)}$ under G. Put $\mathfrak{h}_{(i)}' = \mathfrak{h}_{(i)} \cap \mathfrak{g}_0'$ and $V_i = \bigcup_{x \in G} x\mathfrak{h}_{(i)}'$. Let X be an element $\mathfrak{g}_0'$, and let $\mathfrak{h}_X$ denote the centralizer of X in $\mathfrak{g}_0$. Then since X is regular, $\mathfrak{h}_X$ is a Cartan subalgebra. Hence $\mathfrak{h}_X = x\mathfrak{h}_{(i)}$ for some $x \in G$ and some i $(1 \leqq i \leqq N)$. Therefore $X \in \mathfrak{h}_X \cap \mathfrak{g}_0' \subset V_i$, and this proves that $\mathfrak{g}_0' = V_1 \cup V_2 \cup \cdots \cup V_N$. On the other hand, we know that $\mathfrak{h}_0'$ (and therefore, similarly $\mathfrak{h}_{(i)}'$) has only a finite number of connected components. Since G is connected, it is therefore obvious that the same holds for each V_i. But this implies the statement of the lemma.

LEMMA 29. *Let* $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_N$ *be all the distinct connected components of* $\mathfrak{g}_0'$ *and* c_i *the constant such that* $T' = c_i$ *on* $\mathfrak{g}_i$ $(1 \leqq i \leqq N)$. *Let* Ω *be an open subset of* $\mathfrak{g}_0$ *whose closure is compact. Then there exists an integer* $k \geqq 0$ *such that*

$$\langle \eta^k, \eta^k \rangle T'(f) = \sum_{1 \leqq i \leqq N} c_i \int_{\mathfrak{g}_i} \eta(X)^k f(X; \partial(\eta^k))\, dX$$

for all $f \in C_c^\infty(\Omega)$.

Let τ' denote the distribution on $\mathfrak{g}_0$ given by

$$\tau'(g) = T'(g) - \sum_{1 \leqq i \leqq N} c_i \int_{\mathfrak{g}_i} g(X)\, dX \qquad (g \in C_c^\infty(\mathfrak{g}_0)).$$

Then it is clear that the carrier of τ' is contained in the set of singular points of $\mathfrak{g}_0$. For any $f \in C_c^\infty(\Omega)$, let τ_f denote the distribution on R defined as follows:

$$\tau_f(\beta) = \tau'(\beta(\eta) f) \qquad (\beta \in C_c^\infty(R)),$$

where $\beta(\eta)$ is the function on $\mathfrak{g}_0$ whose value at $X \in \mathfrak{g}_0$ is $\beta(\eta(X))$. It is clear that no point in R other than zero, can lie in the carrier of τ_f. Moreover, since the closure of Ω is compact, τ' is of finite order on Ω (see Schwartz [6(I) pp. 82-83]). Hence there exists an integer k (independent of f) such that the order of τ_f (on R) is $< k$. This implies that $\tau'(\eta^k f) = 0$ for all $f \in C_c^\infty(\Omega)$.

For any $p \in S(\mathfrak{g})$ define $\hat{p}$ as in the proof of Lemma 20. Then obviously, $\hat{\eta}$ coincides on $\mathfrak{h}$ with $(-1)^r \pi^2$ (where r is the degree of π). Therefore it follows from Lemma 9 of [2(h)] that $\hat{\eta} = (-1)^r \eta$, and so

$$(\eta^k \partial(\eta^k) g)^\sim = \partial(\eta^k)(\eta^k \tilde{g}) \qquad (g \in \mathcal{B}(\mathfrak{g}_0))$$

from Lemma 1 of [2(h)]. Hence we conclude from Lemma 26 that

$$T'(\eta^k \partial(\eta^k) g) = \langle \eta^k, \eta^k \rangle T'(g).$$

Now suppose $f \in C_c^\infty(\Omega)$. Then $\tau'(\eta^k \partial(\eta^k) f) = 0$, and therefore

$$\langle \eta^k, \eta^k \rangle T'(f) = T'(\eta^k \partial(\eta^k) f) = \sum_{1 \leqq i \leqq N} c_i \int_{\mathfrak{g}_i} \eta(X)^k f(X; \partial(\eta^k))\, dX.$$

COROLLARY. *If* $c_1, c_2, \cdots, c_N$ *are all equal then* $T' = c_1$ *on* $\mathfrak{g}_0$.

It is sufficient to prove that $T' = c_1$ on Ω. But if $f \in C_c^\infty(\Omega)$, we know from the above lemma that

$$\langle \eta^k, \eta^k \rangle T'(f) = c_1 \int_{\mathfrak{g}_0} \eta(X)^k f(X; \partial(\eta^k))\, dX$$

since $c_1 = c_2 = \cdots = c_N$. Moreover, since η is a homogeneous polynomial of even degree, $\partial(\eta^k)$ is self-adjoint, and therefore

$$\int_{\mathfrak{g}_0} \eta(X)^k f(X; \partial(\eta^k))\, dX = \langle \eta^k, \eta^k \rangle \int_{\mathfrak{g}_0} f(X)\, dX.$$

On the other hand, $\langle \eta^k, \eta^k \rangle \neq 0$ (see the corollary to Lemma 18 of [2(h)]). Therefore

$$T'(f) = c_1 \int_{\mathfrak{g}_0} f(X)\, dX,$$

and this proves that $T' = c_1$ on Ω.

In view of this corollary, our main task now is to prove that the constants $c_1, \cdots, c_N$ of Lemma 29 are all equal. This will be done in another paper. However we note the following result for later use.

LEMMA 30. *Let ξ be any homogeneous polynomial in $I(\mathfrak{g})$ of positive degree. Then $\partial(\xi) T' = 0$.*

Define ξ' as in Lemma 20. Then if $f \in \mathcal{B}(\mathfrak{g}_0)$.

$$T'(\partial(\xi) f) = T((\partial(\xi) f)^{\sim}) = (-1)^m T(\xi' \tilde{f}),$$

where m is the degree of ξ. Since $\xi' \in I(\mathfrak{g})$, it follows from Theorem 3 that $\phi_{\xi' \tilde{f}} = \overline{\xi}' \phi_{\tilde{f}}$, where $\overline{\xi}'$ is the restriction of ξ' on $\mathfrak{h}_0$. Therefore

$$T(\xi' \tilde{f}) = \lim_{\mathfrak{h}_1} \phi_{\tilde{f}}(0; \partial(\pi) \circ \overline{\xi}') = \lim_{\mathfrak{h}_1} \phi_{\tilde{f}}(0; d),$$

where d is the local expression of $\partial(\pi) \circ \overline{\xi}'$ at zero. But since $m \geqq 1$, it follows from Lemma 18 of [2(h)] that $d = 0$. This proves that $\partial(\xi) T' = 0$.

8. Fundamental Cartan subalgebras. Let ω be the Casimir polynomial of $\mathfrak{g}$ and Γ_0 a Cartan subalgebra of $\mathfrak{g}_0$. Put $l_-(\Gamma_0) = \sup_{\Gamma_1} (\dim_R \Gamma_1)$, where Γ_1 runs over all linear subspaces of Γ_0 on which ω is negative-definite. Moreover let $l_- = \sup_{\Gamma_0} l_-(\Gamma_0)$, where Γ_0 runs over all Cartan subalgebras of $\mathfrak{g}_0$. We say that Γ_0 is *fundamental* if $l_-(\Gamma_0) = l_-$. It is obvious that $l_-(x\Gamma_0) = l_-(\Gamma_0)$ $(x \in G)$. Hence if Γ_0 is fundamental, the same holds for any Cartan subalgebra conjugate to Γ_0 under G.

LEMMA 31. $l_- = \operatorname{rank} \mathfrak{k}$.

Let Γ_0 be a fundamental Cartan subalgebra of $\mathfrak{g}_0$. By going over to a conjugate subalgebra, we may assume (see the Corollary of Lemma 1 of [2(e)]) that $\theta(\Gamma_0) = \Gamma_0$. Then since ω is positive-definite on $\mathfrak{p}_0$ and negative-

definite on $\mathfrak{k}_0$, it is clear that $l_- = l_-(\Gamma_0) = \dim_R(\Gamma_0 \cap \mathfrak{k}_0)$. Hence $l_- \leqq \operatorname{rank} \mathfrak{k}$. Conversely, let $\mathfrak{a}_{\mathfrak{k}_0}$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. We extend it to a Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}_0$. Then

$$l_- \geqq l_-(\mathfrak{a}_0) \geqq \dim_R \mathfrak{a}_{\mathfrak{k}_0} = \operatorname{rank} \mathfrak{k}.$$

This proves that $l_- = \operatorname{rank} \mathfrak{k}$.

COROLLARY. *Let Γ_0 be a Cartan subalgebra of $\mathfrak{g}_0$ such that $\theta(\Gamma_0) = \Gamma_0$. Then Γ_0 is fundamental if and only if $\Gamma_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{k}_0$.*

For it is obvious that $l_-(\Gamma_0) = \dim_R(\Gamma_0 \cap \mathfrak{k}_0)$, and therefore from the above lemma, $l_-(\Gamma_0) = l_-$ if and only if $\Gamma_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{k}_0$.

LEMMA 32. *Let $\Gamma_{\mathfrak{k}_0}$ be a maximal abelian subalgebra of $\mathfrak{k}_0$ and Γ_0 the centralizer of $\Gamma_{\mathfrak{k}_0}$ in $\mathfrak{g}_0$. Then Γ_0 is a Cartan subalgebra of $\mathfrak{g}_0$ and $\theta(\Gamma_0) = \Gamma_0$.*

Since $\theta(\Gamma_{\mathfrak{k}_0}) = \Gamma_{\mathfrak{k}_0}$, it is obvious that $\theta(\Gamma_0) = \Gamma_0$. Let $X, Y \in \Gamma_{\mathfrak{p}_0} = \Gamma_0 \cap \mathfrak{p}_0$. Then $Z = [X, Y] \in \Gamma_0 \cap \mathfrak{k}_0 = \Gamma_{\mathfrak{k}_0}$ and $\| Z \|^2 = -B(Z, Z) = B(Y, [X, Z]) = 0$ since $[X, Z] = 0$. This proves that $Z = 0$, and therefore $\Gamma_0 = \Gamma_{\mathfrak{k}_0} + \Gamma_{\mathfrak{p}_0}$ is abelian. Let $\mathfrak{a}_0$ be any Cartan subalgebra of $\mathfrak{g}_0$ containing $\Gamma_{\mathfrak{k}_0}$. Then $\mathfrak{a}_0 \subset \Gamma_0$. But since $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{g}_0$, it follows that $\mathfrak{a}_0 = \Gamma_0$. This proves our assertion.

COROLLARY. *Any two fundamental Cartan subalgebras of $\mathfrak{g}_0$ are conjugate under G.*

Let Γ_1, Γ_2 be two fundamental Cartan subalgebras of $\mathfrak{g}_0$. By replacing them by conjugate subalgebras, we may assume that $\theta(\Gamma_i) = \Gamma_i$ $(i = 1, 2)$. Then from the Corollary to Lemma 31, $\Gamma_i \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{k}_0$. However, since K is compact, any two maximal abelian subalgebras of $\mathfrak{k}_0$ are conjugate under K. Hence $\Gamma_2 \cap \mathfrak{k}_0 = (k\Gamma_1) \cap \mathfrak{k}_0$ for some $k \in K$. But then it follows from Lemma 32 that $\Gamma_2 = k\Gamma_1$.

LEMMA 33. *Let Γ_0 be a Cartan subalgebra of $\mathfrak{g}_0$ and Γ its complexification in $\mathfrak{g}$. Then the following two conditions on Γ_0 are equivalent.*

(1) *Γ is not fundamental.*

(2) *There exists a root α (of $\mathfrak{g}$ with respect to Γ) which takes only real values on Γ_0.*

Again we may assume that $\theta(\Gamma_0) = \Gamma_0$. Since every root α takes only pure imaginary values on $\Gamma_0 \cap \mathfrak{k}_0$, condition (2) is obviously equivalent to the following:

(3) *There exists a root α which vanishes identically on $\Gamma \cap \mathfrak{k}$.*

Now if Γ_0 is fundamental, Γ is exactly the centralizer of $\Gamma \cap \mathfrak{k}$ in $\mathfrak{g}$ (Lemma 32), and therefore no root can vanish identically on $\Gamma \cap \mathfrak{k}$. This proves that (3) implies (1). Conversely, suppose (3) is false. Then no root can vanish identically on $\Gamma \cap \mathfrak{k}$, and therefore it is obvious that Γ is exactly the centralizer of $\Gamma \cap \mathfrak{k}$ in $\mathfrak{g}$. This proves that $\Gamma_0 \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{k}_0$, and so Γ_0 is fundamental. Hence (1) and (3), and therefore also (1) and (2), are equivalent.

We now return to the notation of Section 5. Let Q_+ be the set of those $\alpha \in P$ which take only real values on $\mathfrak{h}_0$. Put $Q' = Q_+ \cup P_-$, and let $W_{Q'}$ be the subgroup of the Weyl group W (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) generated by the Weyl reflexions s_α corresponding to $\alpha \in Q'$.

LEMMA 34. *Suppose $s \in W_{Q'}$. Then $s\mathfrak{h}_0 = \mathfrak{h}_0$ and*

$$\phi_f(sH) = \epsilon(s)\phi_f(H)$$

for $f \in \mathcal{C}(\mathfrak{g}_0)$ and $H \in \mathfrak{h}_0'$.

Let A' be the normalizer of A in G. Then for every $y \in A'$, we get a transformation τ_y of $G^* = G/A$ onto itself given by $\tau_y(x^*) = (xy)^*$ $(x \in G)$. Since A is of finite index in A' (see [2(e), §2]), τ_y^k is the identity for some integer $k \geqq 1$, and from this it follows immediately that the measure dx^* on G^* is invariant under τ_y.

Now from Corollary 2 to Lemma 46 of the Appendix (Section 12) we can choose $y \in G$ such that $yH = sH$ for all $H \in \mathfrak{h}$. Then it is clear that $s\mathfrak{h}_0 = \mathfrak{h}_0$ and $y \in A'$. Therefore if $f \in \mathcal{C}(\mathfrak{g}_0)$,

$$\phi_f(sH) = \pi(sH) \int_{G^*} f(x^*yH)dx^* = \epsilon(s)\pi(H) \int_{G^*} f(x^*H)dx^* = \epsilon(s)\phi_f(H)$$

for $H \in \mathfrak{h}'$.

COROLLARY. *If $\mathfrak{h}_0$ is not fundamental, the distribution T of Lemma 26 is zero.*

We know from Lemma 33 that Q_+ is not empty. Let V_1 denote the closure of $\mathfrak{h}_1$ in $\mathfrak{h}_0$. It follows from Lemma 45 of the Appendix (§12) that we can choose a point $H_0 \in V_1$ and a root $\alpha \in Q_+$ such that $\alpha(H_0) = 0$ while $\beta(H_0) \neq 0$ for any positive root $\beta \neq \alpha$. It is obvious that $tH_0 \in V_1$ for any positive number t. Let s_α denote the Weyl reflexion corresponding to α and for a fixed $f \in \mathcal{C}(\mathfrak{g}_0)$, put

$$\psi_f(H) = \epsilon_+(H)\phi_f(H) \qquad (H \in \mathfrak{h}_0')$$

in the notation of Theorem 3. Since $\mathfrak{h}_\mathfrak{k}$ and $\mathfrak{h}_\mathfrak{p}$ are orthogonal, it is clear

that $s_\alpha\beta = \beta$ for any $\beta \in P_0 \cup P_-$, and therefore $\pi_+{}^{s_\alpha} = -\pi_+$. This shows that $\epsilon_+(s_\alpha H) = -\epsilon_+(H)$, and hence $\psi_f(s_\alpha H) = \psi_f(H)$ from Lemma 34.

Now we extend ψ_f to a function of class C^∞ on $\mathfrak{h}_0''$ (see Theorem 3). Then since $\pi^{s_\alpha} = -\pi$, it is obvious that

$$\psi_f(s_\alpha H; \partial(\pi)) = -\psi_f(H; \partial(\pi)) \qquad (H \in \mathfrak{h}_0'').$$

Therefore if σ_α is the hyperplane in $\mathfrak{h}_0$ consisting of those $H \in \mathfrak{h}_0$ where $\alpha(H) = 0$, it is clear that $\partial(\pi)\psi_f$ is zero on $\sigma_\alpha \cap \mathfrak{h}_0''$. Select a sequence $\{t_k\}_{k\geq 1}$ of positive numbers such that $\operatorname{Lim}_{k\to\infty} t_k = 0$. Then $t_k H_0 \in V_1 \cap \sigma_\alpha \cap \mathfrak{h}_0''$, and so we can choose $H_k \in \mathfrak{h}_1$ such that $\| t_k H_0 - H_k \| \leqq 2^{-k}$ and $|\psi_f(H_k; \partial(\pi))| \leqq 2^{-k}$. Then $H_k \to 0$ and $T(f) = \operatorname{Lim}_{k\to\infty} \phi_f(H_k; \partial(\pi))$. But

$$|\phi_f(H_k; \partial(\pi))| = |\psi_f(H_k; \partial(\pi))| \leqq 2^{-k},$$

and therefore $T(f) = 0$. This proves that $T = 0$.

We shall see in another paper that if $\mathfrak{h}_0$ is fundamental and $\mathfrak{h}_1$ is suitably chosen, then $T \neq 0$. For a special case, this follows from Lemma 41 of Section 11.

9. Closer study of a special case. *In this section we shall assume that* $\mathfrak{h}_0 \subset \mathfrak{k}_0$ and therefore rank $\mathfrak{g}$ = rank $\mathfrak{k}$. Let dY denote the Euclidean measure on $\mathfrak{k}_0$, and, for any $g \in \mathcal{C}(\mathfrak{k}_0)$, put

$$\hat{g}(X) = \int_{\mathfrak{k}_0} \exp\{(-1)^{\frac{1}{2}} B(X, Y)\} g(Y)\, dY \qquad (X \in \mathfrak{g}_0).$$

It is clear that $\hat{g}(Y + Z) = \hat{g}(Y)$ $(Y \in \mathfrak{k}_0, Z \in \mathfrak{p}_0)$ and $\hat{g}$ coincides on $\mathfrak{k}_0$ with the Fourier transform of g. Hence

$$\hat{g}(Y + Z; \partial(p))\,(1 + \| Y + Z + \theta(Y + Z) \|)^m = \hat{g}(Y; \partial(p_\mathfrak{k}))\,(1 + 2\| Y \|)^m.$$

for $p \in S(\mathfrak{g})$ and $m \geqq 0$. (Here $p_\mathfrak{k}$ is the restriction of p on $\mathfrak{k}$). This shows that $\hat{g} \in \mathcal{S}(\mathfrak{g}_0)$ and $g \to \hat{g}$ is a continuous mapping of $\mathcal{C}(\mathfrak{k}_0)$ into $\mathcal{S}(\mathfrak{g}_0)$. Let $\bar{\mathfrak{h}}_0$ be any Cartan subalgebra of $\mathfrak{g}_0$ such that $\theta(\bar{\mathfrak{h}}_0) = \bar{\mathfrak{h}}_0$. We use the notation of Section 6 and define $\bar{\mathfrak{h}}_0'$, $\bar{\pi}$, $\bar{W}$, $\bar{G}$, $d\bar{x}$, etc. as before. Put

$$\bar{\phi}_f(\bar{H}) = \bar{\pi}(\bar{H}) \int_{\bar{G}} f(\bar{x}\bar{H})\, d\bar{x} \qquad (f \in \mathcal{S}(\mathfrak{g}_0), \bar{H} \in \bar{\mathfrak{h}}_0').$$

Then we know from Theorem 3 that $\bar{\phi}_f \in \mathcal{S}(\bar{\mathfrak{h}}_0')$ and $f \to \bar{\phi}_f$ is a continuous mapping of $\mathcal{S}(\mathfrak{g}_0)$ into $\mathcal{S}(\bar{\mathfrak{h}}_0')$. Fix a point $\bar{H}_0$ in $\bar{\mathfrak{h}}_0'$, and put $\sigma_{\bar{H}_0}(g) = \bar{\phi}_{\hat{g}}(\bar{H}_0)$ $(g \in \mathcal{C}(\mathfrak{k}_0))$. Then it is obvious that $\sigma_{\bar{H}_0}$ is a $\mathcal{C}$-distribution on $\mathfrak{k}_0$. Define D and ζ as in Theorem 5 of [2(h)] and p' as in Lemma 20 $(p \in S(\mathfrak{g}))$.

LEMMA 35. *$D\sigma_{\bar{H}_0} = \omega'(\bar{H}_0)\zeta\sigma_{\bar{H}_0}$ where ω is the Casimir polynomial of $\mathfrak{g}$.*

For any $f \in \mathcal{D}(\mathfrak{g}_0)$ and $\alpha \in C_c^\infty(G)$, put

$$f_\alpha(X) = \int_G \alpha(x) f(x^{-1}X)\,dx \qquad (X \in \mathfrak{g}_0).$$

Then since

$$\int_{\bar{G}} |f(\bar{x}\bar{H})|\,d\bar{x} < \infty \qquad (\bar{H} \in \bar{\mathfrak{h}}_0')$$

(Theorem 3), it follows by Fubini's theorem that

$$\bar{\pi}(\bar{H}) \int_{\bar{G}} f_\alpha(\bar{x}\bar{H})\,d\bar{x} = \int_G \alpha(x)\,dx\,\bar{\phi}_f(\bar{H}).$$

Hence

$$\bar{\pi}(\bar{H}_0) \int \hat{g}_\alpha(\bar{x}\bar{H}_0)\,d\bar{x} = \int_G \alpha(x)\,dx\,\sigma_{\bar{H}_0}(g)$$

for $g \in \mathcal{B}(\mathfrak{k}_0)$ and $\alpha \in C_c^\infty(G)$. Now select $\alpha_0 \in C_c^\infty(G)$ such that $\int_G \alpha_0(x)\,dx = 1$. Then

$$\sigma_{\bar{H}_0}(g) = \bar{\pi}(\bar{H}_0) \int \hat{g}_{\alpha_0}(\bar{x}\bar{H}_0)\,d\bar{x} \qquad (g \in \mathcal{B}(\mathfrak{k}_0)),$$

and since $\omega' \in I(\mathfrak{g})$, it is obvious that

$$\omega'(\bar{H}_0)\sigma_{\bar{H}_0}(g) = \bar{\pi}(\bar{H}_0) \int_G \omega'(\bar{x}\bar{H}_0)\hat{g}_{\alpha_0}(\bar{x}\bar{H}_0)\,d\bar{x}.$$

Put

$$\Lambda_X(x:Y) = \exp\{(-1)^{\frac{1}{2}} B(X, xY)\} \qquad (x \in G, Y \in \mathfrak{k}_0)$$

for a fixed $X \in \mathfrak{g}_0$. From Lemma 24 of [2(h)] we know that $\Gamma_Y(\gamma_\omega(Y)) = \zeta(Y)\omega$ $(Y \in \mathfrak{k}_0)$ and

$$\gamma_\omega(Y) = \sum_{1 \leqq i \leqq r} q_i(Y)\,(b_i \mathbf{X} 1) + 1 \mathbf{X} D_Y \qquad (Y \in \mathfrak{k}_0).$$

Here D_Y is the local expression of D at Y, the q_i are polynomial functions on $\mathfrak{k}$ and the b_i are homogeneous elements in $\mathfrak{P}$ of order 2 $(1 \leqq i \leqq r)$. Let $\lambda_X(Z) = \exp\{(-1)^{\frac{1}{2}} B(X, Z)\}$ $(Z \in \mathfrak{g}_0)$ for a fixed $X \in \mathfrak{g}_0$. Then from Lemma 3 of [2(h)] it is clear that

$$\zeta(Y)\lambda_X(xY; \partial(\omega)) = \sum_{1 \leqq i \leqq r} q_i(Y)\Lambda_X(x; b_i : Y) + \Lambda_X(x : Y; D)$$

and therefore

$$\int_{G \times \mathfrak{k}_0} \alpha_0(x)\zeta(Y)\lambda_X(xY; \partial(\omega))\,g(Y)\,dx\,dY$$

$$= \sum_{1 \leqq i \leqq r} \int_{G \times \mathfrak{k}_0} \alpha_0(x; b_i^*)\,q_i(Y)\,g(Y)\lambda_X(xY)\,dx\,dY$$

$$+ \int_{G \times \mathfrak{k}_0} \alpha_0(x)\,g(Y; D^*)\lambda_X(xY)\,dx\,dY,$$

where the star denotes the adjoint of a differential operator. But

$$\lambda_X(Z;\partial(\omega)) = \omega'(X)\lambda_X(Z) \qquad (Z \in \mathfrak{g}_0).$$

Therefore[11] if $g_i = q_i g$,

$$\omega'(\zeta g)\hat{}_{a_0} = \sum_{1 \leqq i \leqq r} (\hat{g}_i)_{b_i^* a_0} + (D^* g)\hat{}_{a_0}.$$

Moreover since the b_i are homogenous of degree 2, it is obvious that $b_i = b_i^* \in \mathfrak{Bg}$ and so $\int \alpha_0(x; b_i^*)\,dx = 0$. So we conclude that

$$\omega'(\bar{H}_0)\sigma_{\bar{H}_0}(\zeta g) = \sigma_{\bar{H}_0}(D^* g),$$

and this proves that $D\sigma_{\bar{H}_0} = \omega'(\bar{H}_0)\zeta\sigma_{\bar{H}_0}$.

Define the mapping $f \to \psi_f$ of $\mathcal{B}(\mathfrak{h}_0)$ into $J(\mathfrak{k}_0)$ as in Lemma 27 of [2(h)] and put $\Phi_{\bar{H}_0}(f) = \sigma_{\bar{H}_0}(\zeta\psi_f)$ $(f \in \mathcal{B}(\mathfrak{h}_0))$. Then $\Phi_{\bar{H}_0}$ is a $\mathcal{B}$-distribution, which, from Theorem 5 of [2(h)], coincides with an analytic function on $\mathfrak{h}_0$. Our first object is to show that $\Phi_{\bar{H}_0} = 0$ if $\bar{\mathfrak{h}}_0$ is not conjugate to $\mathfrak{h}_0$ under G. For this we need a few lemmas.

LEMMA 36. *For any $f \in C_c^\infty(G \times \mathfrak{h}_0')$, put*

$$\hat{f}(X) = \int_{G \times \mathfrak{h}_0'} \exp\{(-1)^{\frac{1}{2}} B(X, xH)\} f(x, H)\,dx\,dH \qquad (X \in \mathfrak{g}_0),$$

where dH is the regular Euclidean measure on $\mathfrak{h}_0'$. Then $\hat{f} \in \mathcal{B}(\mathfrak{g}_0)$ and $f \to \hat{f}$ is a continuous mapping of $C_c^\infty(G \times \mathfrak{h}_0')$ into $\mathcal{B}(\mathfrak{g}_0)$.

Put $G^* = G/A$ as before, and let H_0 be any point in $\mathfrak{h}_0'$. Then we can select (see Section 6) an open neighborhood U of H_0 in $\mathfrak{h}_0'$ such that the mapping $\phi : (x^*, H) \to x^* H$ of $G^* \times U$ into $\mathfrak{g}_0$ is univalent and regular and, if the Euclidean measure dX on $\mathfrak{g}_0$ is suitably normalized, $dX = |\pi(H)|^2 dx^* dH$ $(X = x^* H, x^* \in G^*, H \in U)$. Then $N = \phi(G^* \times U)$ is an open subset of $\mathfrak{g}_0$ and ϕ defines an analytic isomorphism of $G^* \times U$ with N. Therefore, for any $f \in C_c^\infty(G \times U)$, we can define a function $F_f \in C_c^\infty(N)$ by

$$F_f(x^* H) = |\pi(H)|^2 \int_A f(xa, H)\,da \qquad (x \in G, H \in U).$$

Here da is the Haar measure on A normalized so that

$$\int_G g(x)\,dx = \int_{G^*} dx^* \int_A g(xa)\,da \qquad (g \in C_c(G)).$$

[11] Here $(h)\hat{}_a = (\hat{h})_a$ for $h \in \mathcal{B}(\mathfrak{k}_0)$ and $a \in C_c^\infty(G)$.

It is now clear that

$$\hat{f}(X') = \int_{\mathfrak{g}_0} \exp\{(-1)^{\frac{1}{2}} B(X', X)\} F_f(X) dX \qquad (X' \in \mathfrak{g}_0),$$

and therefore $\hat{f}$ is the Fourier transform of F_f. This proves that $\hat{f} \in \mathcal{B}(\mathfrak{g}_0)$. Moreover, since ϕ is an analytic isomorphism, the mapping $f \to F_f$ of $C_c^\infty(G \times U)$ into $C_c^\infty(N)$ is obviously continuous, and so the same holds for the mapping $f \to \hat{f}$ of $C_c^\infty(G \times U)$ into $\mathcal{B}(\mathfrak{g}_0)$. The statement of the lemma now follows from the fact that any compact subset of $\mathfrak{h}_0'$ can be covered by a finite number of sets of the type U.

For a fixed $\bar{H} \in \bar{\mathfrak{h}}_0'$, put

$$T_{\bar{H}}(g) = \bar{\pi}(\bar{H}) \int_{\bar{G}} g(\bar{x}\bar{H}) d\bar{x} \qquad (g \in \mathcal{B}(\mathfrak{g}_0)).$$

Then from Theorem 3, $T_{\bar{H}}$ is a $\mathcal{B}$-distribution on $\mathfrak{g}_0$. Put $\Phi_f(\bar{H}) = T_{\bar{H}}(\hat{f})$ for $f \in C_c^\infty(G \times \mathfrak{h}_0')$. Then it follows from Lemma 36 that for fixed $\bar{H}$ the mapping $f \to \Phi_f(\bar{H})$ is a distribution on $G \times \mathfrak{h}_0'$. If $\alpha \in C_c^\infty(G)$ and $\beta \in C_c^\infty(\mathfrak{h}_0')$, we denote by $\alpha \times \beta$ the function $\gamma \in C_c^\infty(G \times \mathfrak{h}_0')$ given by $\gamma(x, H) = \alpha(x)\beta(H)$ $(x \in G, H \in \mathfrak{h}_0')$.

LEMMA 37. *For any $\bar{H} \in \bar{\mathfrak{h}}_0'$, there exists a distribution $\tau_{\bar{H}}'$ on $\mathfrak{h}_0'$ such that*

$$\Phi_{\alpha \times \beta}(\bar{H}) = \int_G \alpha(x) dx \, \tau_{\bar{H}}'(\beta)$$

for $\alpha \in C_c^\infty(G)$ and $\beta \in C_c^\infty(\mathfrak{h}_0')$.

It is obvious that for a fixed β, the mapping $\alpha \to \Phi_{\alpha \times \beta}(\bar{H})$ is a distribution on G which is invariant under left translations. Therefore from Lemma 36 of [2(e)], there exists a constant c_β such that

$$\Phi_{\alpha \times \beta}(\bar{H}) = c_\beta \int_G \alpha(x) dx \qquad (\alpha \in C_c^\infty(G)).$$

Select $\alpha_0 \in C_c^\infty(G)$ such that $\int_G \alpha_0(x) dx = 1$. Then $c_\beta = \Phi_{\alpha_0 \times \beta}(\bar{H})$. But it is evident that the mapping $\tau_{\bar{H}}' : \beta \to \Phi_{\alpha_0 \times \beta}(\bar{H})$ $(\beta \in C_c^\infty(\mathfrak{h}_0'))$ is a distribution on $\mathfrak{h}_0'$. Hence

$$\Phi_{\alpha \times \beta}(\bar{H}) = \int_G \alpha(x) dx \, \tau_{\bar{H}}'(\beta) \qquad (\alpha \in C_c^\infty(G), \beta \in C_c^\infty(\mathfrak{h}_0')).$$

Put $\tau_{\bar{H}} = \pi \tau_{\bar{H}}'$ $(\bar{H} \in \bar{\mathfrak{h}}_0')$.

LEMMA 38. *For any* $p \in I(\mathfrak{g})$ *and* $\bar{H} \in \bar{\mathfrak{h}}_0'$,

$$\partial(\bar{p})\tau_{\bar{H}} = p'(\bar{H})\tau_{\bar{H}}$$

where $\bar{p}$ *denotes the restriction of* p *on* $\mathfrak{h}$.

We use the notation of the proof of Lemma 35. It is obvious that

$$\lambda_X(Z;\partial(q)) = q'(X)\lambda_X(Z) \qquad (X, Z \in \mathfrak{g}_0; q \in S(\mathfrak{g})).$$

Select $\alpha \in C_c^\infty(G)$ such that $\int \alpha(x)dx = 1$, and fix $\beta \in C_c^\infty(\mathfrak{h}_0')$. Then if if $p \in I(\mathfrak{g})$, it follows from Lemmas 5 and 8 of [2(h)] that

$$p'(X)\lambda_X(xH) = \lambda_X(xH;\partial(p))$$
$$= \sum_{1 \leqq i \leqq N} a_i(H)\Lambda_X(x;b_i:H;\partial(u_i)) + \pi(H)^{-1}\Lambda_X(x:H;\partial(\bar{p})\circ\pi)$$

for $x \in G, H \in \mathfrak{h}_0'$ and $X \in \mathfrak{g}_0$. Here $u_i \in S(\mathfrak{h})$, $b_i \in \mathfrak{B}\mathfrak{g}$ and the a_i are analytic functions on $\mathfrak{h}_0'$. Hence if $\gamma = \alpha \times \beta$,

$$p'(X)\hat{\gamma}(X) = \sum_{1 \leqq i \leqq N} \int \alpha_i(x)\beta_i(H)\lambda_X(xH)dxdH + \int \alpha(x)\beta'(H)\lambda_X(xH)dxdH$$

where $\alpha_i = b_i{}^*\alpha$, $\beta_i = (\partial(u_i))^*(a_i\beta)$ $(1 \leqq i \leqq N)$ and $\beta' = (\pi^{-1}\partial(\bar{p})\circ\pi)^*\beta$. (As usual the star denotes the adjoint of a differential operator). But then

$$\int \alpha_i(x)dx = \int \alpha(x;b_i{}^*)dx = 0$$

and so it follows from Lemma 37 that

$$T_{\bar{H}}(p'\hat{\gamma}) = \tau_{\bar{H}}'(\beta').$$

Moreover $p' \in I(\mathfrak{g})$ and therefore it is obvious that

$$T_{\bar{H}}(p'\gamma) = p'(\bar{H})T_{\bar{H}}(\hat{\gamma}) = p'(\bar{H})\tau_{\bar{H}}'(\beta).$$

This proves that

$$\tau_{\bar{H}}'(\beta') = p'(\bar{H})\tau_{\bar{H}}'(\beta).$$

But $\beta' = (\pi^{-1}\partial(\bar{p})\circ\pi)^*\beta$. Hence if we replace β by $\pi\beta$, we get

$$\tau_{\bar{H}}((\partial(\bar{p}))^*\beta) = p'(\bar{H})\tau_{\bar{H}}(\beta),$$

and so the lemma is proved.

Now normalize the Euclidean measure dY on $\mathfrak{k}_0$ in such a way that

$$\int_{\mathfrak{k}_0} g(Y)dY = \int_{\mathfrak{h}_0} |\pi_-(H)|^2 dH \int_K g(kH)dK \qquad (g \in C_c(\mathfrak{k}_0)).$$

Here $\pi_- = \prod_{\alpha \in P_-} \alpha$ and dk is the Haar measure on K ($\int_K dk = 1$). It is easily seen that $\zeta(H)|\pi_-(H)|^2 = (-1)^r \pi(H)^2$ $(H \in \mathfrak{h}_0)$, where r is the degree of π. Therefore it follows that if $f \in \mathcal{B}(\mathfrak{h}_0)$ and $X \in \mathfrak{g}_0$, then

$$\int_{\mathfrak{t}_0} \exp\{(-1)^{\frac{1}{2}} B(X,Y)\}\zeta(Y)\psi_f(Y)\,dY$$
$$= (-1)^r \int_{K \times \mathfrak{b}_0} \exp\{(-1)^{\frac{1}{2}} B(X, kH)\}\pi(H) \sum_{s \in W} \epsilon(s) f(sH)\,dk\,dH.$$

Now select $\beta \in C_c^\infty(\mathfrak{h}_0')$ and $\alpha \in C_c^\infty(G)$ ($\int \alpha(x)\,dx = 1$), and put [12]

$$\alpha_1(x) = \int_K \alpha(xk)\,dk, \qquad \beta_1 = \pi \sum_{s \in W} \epsilon(s)\beta^s$$

and $\gamma = \alpha_1 \times \beta_1$. Then $T_{\bar{H}}(\hat{\gamma}) = \Phi_{\alpha_1 \times \beta_1}(\bar{H}) = \tau_{\bar{H}}'(\beta_1)$ since $\int \alpha_1(x)dx = 1$. On the other hand, if $g = \zeta\psi_\beta$,

$$\hat{\gamma}(X) = \int \exp\{(-1)^{\frac{1}{2}} B(X, xkH)\}\alpha(x)\beta_1(H)\,dx\,dk\,dH$$
$$= (-1)^r \hat{g}_a(X) \qquad (X \in \mathfrak{g}_0)$$

in the notation of the proof of Lemma 35. Therefore

$$\tau_{\bar{H}}'(\beta_1) = T_{\bar{H}}(\hat{\gamma}) = (-1)^r \sigma_{\bar{H}}(g) = (-1)^r \Phi_{\bar{H}}(\beta),$$

and so we get the following result.

LEMMA 39. *If the Euclidean measure on $\mathfrak{t}_0$ is suitably normalized, we have the relation*

$$\Phi_{\bar{H}}(\beta) = (-1)^r \sum_{s \in W} \epsilon(s)\tau_{\bar{H}}(\beta^s)$$

for all $\beta \in C_c^\infty(\mathfrak{h}_0')$ and $\bar{H} \in \bar{\mathfrak{h}}_0'$.

COROLLARY 1. *$\partial(\tilde{p})\Phi_{\bar{H}} = p'(\bar{H})\Phi_{\bar{H}}$ for $p \in I(\mathfrak{g})$ and $\bar{H} \in \bar{\mathfrak{h}}_0'$.*

We know from Lemma 9 of [2(h)] that $\tilde{p}$ is invariant under W. Therefore

$$(\partial(\tilde{p}))^*\beta^s = ((\partial(\tilde{p}))^*\beta)^s \qquad (s \in W, \beta \in C_c^\infty(\mathfrak{h}_0'))$$

and

$$\Phi_{\bar{H}}((\partial(\tilde{p}))^*\beta) = (-1)^r \sum_{s \in W} \epsilon(s)\tau_{\bar{H}}((\partial(\tilde{p}))^*\beta^s)$$
$$= (-1)^r p'(\bar{H}) \sum_{s \in W} \epsilon(s)\tau_{\bar{H}}(\beta^s) = p'(\bar{H})\Phi_{\bar{H}}(\beta)$$

[12] As usual β^s $(s \in W)$ is defined by $\beta^s(H) = \beta(s^{-1}H)$ $(H \in \mathfrak{h}_0)$.

from Lemma 38. This proves that $\partial(\tilde{p})\Phi_{\bar{H}} - p'(\bar{H})\Phi_{\bar{H}}$ is zero on $\mathfrak{h}_0'$. But we have seen that $\Phi_{\bar{H}}$ is an analytic function on $\mathfrak{h}_0$. Therefore $\partial(\tilde{p})\Phi_{\bar{H}} = p'(\bar{H})\Phi_{\bar{H}}$ on $\mathfrak{h}_0$.

COROLLARY 2. *Suppose $\mathfrak{h}_0$ is not conjugate to $\bar{\mathfrak{h}}_0$ under G. Then $\Phi_{\bar{H}} = 0$ for every $\bar{H} \in \bar{\mathfrak{h}}_0'$.*

Fix $\bar{H} \in \bar{\mathfrak{h}}_0'$ and consider the $\mathcal{C}$-distribution $\tilde{\Phi}: f \to \Phi_{\bar{H}}(f')$ $(f \in \mathcal{C}(\mathfrak{h}_0))$, where

$$f'(H') = \int_{\mathfrak{h}_0} \exp\{-(-1)^{\frac{1}{2}} B(H', H)\} f(H)\, dH \qquad (H' \in \mathfrak{h}_0).$$

Then it follows from Corollary 1 above that $\tilde{p}\tilde{\Phi} = p(\bar{H})\tilde{\Phi}$ $(p \in I(\mathfrak{g}))$, and therefore if H_0 is any point lying in the carrier of $\tilde{\Phi}$, it is obvious that $p(H_0) = p(\bar{H})$ for all $p \in I(\mathfrak{g})$. Let λ be an indeterminate and I the identity mapping of $\mathfrak{g}_0$. Then

$$\det(\lambda I - \operatorname{ad} X) = \sum_{l \leqq m \leqq n} \lambda^m p_m(X) \qquad (X \in \mathfrak{g}),$$

where $p_m \in I(\mathfrak{g})$. Let $\bar{\mathfrak{h}}$ be the complexification of $\bar{\mathfrak{h}}_0$ in $\mathfrak{g}$ and $\bar{P}$ the set of all positive roots (under some fixed order) of $\mathfrak{g}$ with respect to $\bar{\mathfrak{h}}$. Then

$$\det(\lambda I - \operatorname{ad} H_0) = (-1)^r \lambda^l \prod_{\alpha \in P} (\lambda - \alpha(H_0))^2$$

and similarly,

$$\det(\lambda I - \operatorname{ad} \bar{H}) = (-1)^r \lambda^l \prod_{\bar{\alpha} \in \bar{P}} (\lambda - \bar{\alpha}(\bar{H}))^2.$$

On the other hand, we know from the Corollary to Lemma 32 that $\bar{\mathfrak{h}}_0$ is not a fundamental Cartan subalgebra of $\mathfrak{g}_0$. Therefore from Lemma 33, there exists a root $\bar{\alpha}_0 \in \bar{P}$ which takes only real values on $\bar{\mathfrak{h}}_0$. Then $\bar{\alpha}_0(\bar{H}) \neq 0$ since $\bar{H} \in \bar{\mathfrak{h}}_0'$. On the other hand, every root $\alpha \in P$ takes only pure imaginary values on $\mathfrak{h}_0$ since $\mathfrak{h}_0 \subset \mathfrak{k}_0$. This proves that

$$\det(\lambda I - \operatorname{ad} H_0) \neq \det(\lambda I - \operatorname{ad} \bar{H}),$$

and therefore $p_m(H_0) \neq p_m(\bar{H})$ for some m $(l \leqq m \leqq n)$. This shows that the carrier of $\tilde{\Phi}$ is empty, and therefore since $\tilde{\Phi}$ is the Fourier transform of $\Phi_{\bar{H}}$ (see Schwartz [6(II)]), we conclude that $\Phi_{\bar{H}} = 0$.

Let us now consider the case when $\bar{\mathfrak{h}}_0 = \mathfrak{h}_0$ and denote by $\Phi(H_0 : H)$ the value of the analytic function Φ_{H_0} at H $(H_0 \in \mathfrak{h}_0', H \in \mathfrak{h}_0)$.

LEMMA 40. *There exists a locally constant function*[13] *c on $\mathfrak{h}_0'$ such that*

$$\Phi(H_0 : H) = c(H_0) \sum_{s \in W} \epsilon(s) \exp\{(-1)^{\frac{1}{2}} B(sH_0, H)\}$$

for $H_0 \in \mathfrak{h}_0'$ and $H \in \mathfrak{h}_0$.

[13] This means that c is constant on some neighborhood of every point in $\mathfrak{h}_0'$ and therefore also on each connected component of $\mathfrak{h}_0'$.

Fix $H_0 \in \mathfrak{h}_0'$. Then it follows from Corollary 1 of Lemma 39 and the Corollary to Lemma 13 of [2(h)] that

$$\Phi(H_0 : H) = \sum_{s \in W} c_s \exp\{(-1)^{\frac{1}{2}} B(sH_0, H)\} \qquad (H \in \mathfrak{h}_0),$$

where the c_s are uniquely determined constants. On the other hand, it is obvious from the definition of Φ_{H_0} that $\Phi_{H_0}(\beta^s) = \epsilon(s)\Phi_{H_0}(\beta)$ $(\beta \in \mathcal{E}(\mathfrak{h}_0), s \in W)$. Hence $\Phi(H_0 : sH) = \epsilon(s)\Phi(H_0 : H)$. Therefore, in view of the uniqueness of c_s, we conclude that $\epsilon(s)c_s$ $(s \in W)$ are all equal. This proves that

$$\Phi(H_0 : H) = c(H_0) \sum_{s \in W} \epsilon(s) \exp\{(-1)^{\frac{1}{2}} B(sH_0, H)\} \quad (H_0 \in \mathfrak{h}_0', H \in \mathfrak{h}_0),$$

where c is a function on $\mathfrak{h}_0'$.

Now select $\alpha \in C_c^\infty(G)$ such that $\int \alpha(x)dx = 1$, and for $\beta \in C_c^\infty(\mathfrak{h}_0')$, put $\gamma = \alpha \times \pi_l\beta$. Then $\tau_{H_0}(\beta) = T_{H_0}(\hat{\gamma}) = \phi_{\hat{\gamma}}(H_0)$ in the notations of Lemma 36 and Theorem 3. On the other hand,

$$\Phi_{H_0}(\beta) = (-1)^r \sum_{s \in W} \epsilon(s)\tau_{H_0}(\beta^s)$$

from Lemma 39. Therefore

$$c(H) \sum_{s \in W} \epsilon(s)\tilde{\beta}(sH) = (-1)^r \sum_{s \in W} \epsilon(s)\phi_{\hat{\gamma}_s}(H)$$

$(H \in \mathfrak{h}_0', \beta \in C_c^\infty(\mathfrak{h}_0'))$, where $\tilde{\beta}$ is the Fourier transform of β and $\gamma_s = \alpha \times \pi\beta^s$. Since $C_c^\infty(\mathfrak{h}_0')$ is dense in $\mathcal{E}(\mathfrak{h}_0)$ under the norm

$$\| \beta_1 \| = \int_{\mathfrak{h}_0} |\beta_1(H)| \, dH \qquad (\beta_1 \in \mathcal{E}(\mathfrak{h}_0)),$$

it follows without difficulty that, for a given $H_0 \in \mathfrak{h}_0'$, we can select $\beta \in C_c^\infty(\mathfrak{h}_0')$ such that $\sum_{s \in W} \epsilon(s)\tilde{\beta}(sH_0) \neq 0$. Hence, in view of the fact that $\hat{\gamma}_s \in \mathcal{E}(\mathfrak{g}_0)$ (Lemma 36) and ϕ_g is of class C^∞ on $\mathfrak{h}_0'$ for $g \in \mathcal{E}(\mathfrak{g}_0)$ (Theorem 3), we conclude that c is of class C^∞ on $\mathfrak{h}_0'$. Now suppose $p \in I(\mathfrak{g})$. Then it is evident that

$$\gamma(X ; \partial(p)) = \int \exp\{(-1)^{\frac{1}{2}} B(X, xH)\}\alpha(x)p'(H)\pi(H)\beta(H)dxdH \quad (X \in \mathfrak{g}_0).$$

Hence if $\beta' = \bar{p}'\beta$ and $\gamma' = \alpha \times \pi\beta'$, we get $\partial(p)\hat{\gamma} = \hat{\gamma}'$, and therefore

$$\tau_{H_0}(\beta') = \phi_{\partial(p)\hat{\gamma}}(H_0) = \phi_{\hat{\gamma}}(H_0 ; \partial(\bar{p}))$$

from Theorem 3. Moreover $(\beta')^s = \bar{p}'\beta^s$ $(s \in W)$ since $\bar{p}'$ is invariant under W. Therefore

$$\Phi_{H_0}(\beta') = (-1)^r \sum_{s \in W} \epsilon(s) \tau_{H_0}((\beta')^s) = (-1)^r \sum_{s \in W} \epsilon(s) \phi\hat{}_{\gamma_s}(H_0; \partial(\bar{p}))$$

$$= \int_{\mathfrak{h}_0} \Phi(H_0; \partial(\bar{p}) : H) \beta(H) dH.$$

Since β is an arbitrary element in $C_c^\infty(\mathfrak{h}_0')$, we conclude that

$$\Phi(H_0; \partial(\bar{p}) : H) = p'(H) \Phi(H_0 : H) \qquad (H_0 \in \mathfrak{h}_0', H \in \mathfrak{h}_0).$$

But now it follows from Lemma 23 that c is locally constant.

COROLLARY. *There exists a complex number c_0 such that*

$$\sum_{s \in W} \Phi(sH' : H) = c_0 \sum_{s \in W} \epsilon(s) \exp\{(-1)^{\frac{1}{2}} B(sH', H)\}$$

for all $H' \in \mathfrak{h}_0'$ and $H \in \mathfrak{h}_0$.

Let $\mathfrak{h}_1$ be a connected component of $\mathfrak{h}_0'$. Then (see Lemma 44 of the Appendix, Section 12) $\mathfrak{h}_0' = \bigcup_{s \in W} s\mathfrak{h}_1$, and so it is evident that $c_0 = \sum_{s \in W} c(sH')$ $(H' \in \mathfrak{h}_0')$ is independent of H'. From this our assertion follows immediately.

The main results of this section can now be summarized in the following theorem.

THEOREM 4. *Let $\mathfrak{h}_0$ and $\bar{\mathfrak{h}}_0$ be two Cartan subalgebras of $\mathfrak{g}_0$. We assume that $\mathfrak{h}_0 \subset \mathfrak{k}_0$ and*[14] *$\theta(\bar{\mathfrak{h}}_0) = \bar{\mathfrak{h}}_0$. For any $f \in \mathcal{D}(\mathfrak{h}_0)$, put*

$$\hat{f}(X) = \int_{K \times \mathfrak{h}_0} \exp\{(-1)^{\frac{1}{2}} B(X, kH)\} \pi(H)^2 \sum_{s \in W} f(sH) dk dH \qquad (X \in \mathfrak{g}_0).$$

Then the integral

$$\phi\hat{}_f(\bar{H}) = \bar{\pi}(\bar{H}) \int_{\bar{G}} \hat{f}(\bar{x}\bar{H}) d\bar{x}$$

is convergent for $\bar{H} \in \bar{\mathfrak{h}}_0'$. Moreover in case $\bar{\mathfrak{h}}_0$ is not conjugate to $\mathfrak{h}_0$ under G, $\phi\hat{}_f = 0$ on $\bar{\mathfrak{h}}_0'$. On the other hand, if $\bar{\mathfrak{h}}_0 = \mathfrak{h}_0$, there exists a complex number c_0 such that

$$\sum_{s \in W} \epsilon(s) \phi\hat{}_f(sH') = c_0 \int_{\mathfrak{h}_0} \sum_{s \in W} \epsilon(s) \exp\{(-1)^{\frac{1}{2}} B(H', sH)\} \pi(H) f(H) dH$$

for $H' \in \mathfrak{h}_0'$ and $f \in \mathcal{D}(\mathfrak{h}_0)$.

[14] In view of the corollary to Lemma 1 of [2(e)], our assumption that $\theta(\bar{\mathfrak{h}}_0) = \bar{\mathfrak{h}}_0$ is obviously unnecessary. We make it here only for convenience.

It is evident that $\phi_f^{\wedge}(\bar{H}) = (-1)^r \Phi_{\bar{H}}(\pi f)$ $(f \in \mathcal{B}(\mathfrak{h}_0),\ \bar{H} \in \bar{\mathfrak{h}}_0')$ in our earlier notation. Therefore, from Corollary 2 to Lemma 39, $\phi_f^{\wedge} = 0$ if $\bar{\mathfrak{h}}_0$ is not conjugate to $\mathfrak{h}_0$ On the other hand, if $\bar{\mathfrak{h}}_0 = \mathfrak{h}_0$, our assertion follows from the Corollary to Lemma 40. We shall prove in Section 11 that $c_0 \neq 0$.

10. Some work of de Rham. Let E be a Euclidean space over R of dimension $n \geqq 3$, and let p, q be two nonnegative integers such that $p + q = n$. Let $(x_1, \cdots, x_n)$ be a Cartesian system of coordinates on E and put $y_i = x_{p+i}$ $(1 \leqq i \leqq q)$. We consider the differential operator

$$\square = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \left(\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_q^2} \right)$$

on E. Let dX denote the Euclidean measure on E so normalized that $dX = dx_1 \cdots dx_n$. Then every locally summable function ξ on E defines, as usual, a distribution

$$f \to \int_E f \xi dX \qquad (f \in C_c^{\infty}(E))$$

which will also be denoted by ξ. Let δ denote the Dirac distribution given by $\delta(f) = f(0)$ and put $u = x_1^2 + \cdots + x_p^2 - (y_1^2 + \cdots + y_q^2)$. Then de Rham has shown [15] that there exists a function ξ on R with the following properties. Put $\Xi(X) = \xi(u(X))$ $(X \in E)$. Then Ξ is a locally summable function on E and

$$\square^{[n/2]} \Xi = \delta$$

in the sense of distribution theory. (Here $[t]$ denotes the largest integer not exceeding $t \in R$). The actual expressions for ξ in the various possible cases are given below in terms of the Heaviside function $Y(t)$ which is 0 or 1 according as $t \leqq 0$ or $t > 0$ $(t \in R)$. There are four cases.[16]

(1) If $n \equiv p \equiv 1 \bmod 2$,

$$\xi(t) = (-1)^{(p-1)/2} \{ \pi^{\frac{1}{2}(n-2)} 4^{n-2} \Gamma((n-2)/2) \}^{-1} Y(t) t^{-\frac{1}{2}}.$$

(2) If $n \equiv q \equiv 1 \bmod 2$,

$$\xi(t) = (-1)^{p/2} \{ \pi^{\frac{1}{2}(n-2)} 4^{n-2} \Gamma((n-2)/2) \}^{-1} Y(-t) |t|^{-\frac{1}{2}}.$$

(3) If $n \equiv p \equiv 0 \bmod 2$,

$$\xi(t) = (-1)^{(p-2)/2} \{ (4\pi)^{n/2} \Gamma(n/2) \}^{-1} \log |t|.$$

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[15] I am grateful to Professor de Rham for communicating his results to me, which, so far as I know, have not yet been published.

[16] Here Γ stands for the classical Gamma function.

(4) If $p \equiv q \equiv 1 \bmod 2$,

$$\xi(t) = (-1)^{(p-1)/2}\{(4\pi)^{(n-2)/2}\,\Gamma((n-2)/2)\}^{-1}\,Y(t).$$

Put $P = x_1^2 + \cdots + x_p^2$, $Q = y_1^2 + \cdots, y_q^2$. Then a simple calculation shows that

$$c = \int_E |\,\Xi\,|(1+P)^{-m}(1+Q)^{-m}\,dX < \infty$$

if m is a sufficiently large positive integer. Hence

$$\int |\,\Xi f\,|\,dX \leqq c \sup_{X \in E} |(1+P(X))^m (1+Q(X))^m f(X)|$$

for $f \in \mathcal{B}(E)$. This proves that Ξ can be regarded as a $\mathcal{B}$-distribution. Therefore, since $\square$ is obviously self-adjoint, we get

$$f(0) = \int_E \Xi \square^{[n/2]} f\,dX$$

for any $f \in \mathcal{B}(E)$.

11. An application of the above formula. Let ω be the Casimir polynomial of $\mathfrak{g}$ and put $\square = \partial(\omega)$. Then it is clear from the results of Section 10 that there exists a function ξ on R such that $\Xi(X) = \xi(\omega(X))$ $(X \in \mathfrak{g}_0)$ is a locally summable function on $\mathfrak{g}_0$ and

$$f(0) = \int_{\mathfrak{g}_0} \Xi \partial(\omega^m) f\,dX \qquad (f \in \mathcal{B}(\mathfrak{g}_0)^\sim).$$

Here $m = [n/2]$, $n = \dim_R \mathfrak{g}_0$ and dX is the regular Euclidean measure on $\mathfrak{g}_0$. (Since $\mathfrak{g}_0$ is semisimple it is obvious that $n \geqq 3$). We shall apply this formula to prove that the constant c_0 of Theorem 4 is not zero. So let us assume that $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ which is contained in $\mathfrak{k}_0$. Select a finite set of Cartan subalgebras $\mathfrak{h}_0 = \mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_N$ of $\mathfrak{g}_0$ such that $\theta(\mathfrak{h}_i) = \mathfrak{h}_i$ $(1 \leqq i \leqq N)$ and every Cartan subalgebra of $\mathfrak{g}_0$ is conjugate under G to $\mathfrak{h}_i$ for exactly one i $(1 \leqq i \leqq N)$. We know (see the Corollary to Lemma 2 of [2(e)]) that this is possible. Let $(\mathfrak{h}_i)_c$ denote the complexification of $\mathfrak{h}_i$ in $\mathfrak{g}$ and P_i the set of all positive roots of $\mathfrak{g}$ with respect to $(\mathfrak{h}_i)_c$ under some fixed order. (We assume that $P_1 = P$.) Put $\pi_i = \prod_{\alpha \in P_i} \alpha$ and let W_i denote the Weyl group of $\mathfrak{g}$ with respect to $(\mathfrak{h}_i)_c$. Consider the Cartan subgroup A_i of G corresponding to $\mathfrak{h}_i$ and the normalizer A_i' of A_i

in G. Then[17] the factor group $W_i' = A_i'/A_i$ can be regarded as a subgroup of W_i. Put $G_i^* = G/A_i$, and define x^*H $(x^* \in G_i^*, H \in \mathfrak{h}_i)$ in the usual way. We have seen in Section 8 that W_i' operates on G_i^* on the right. Let $\mathfrak{g}_0'$ be the set of regular elements of $\mathfrak{g}_0$, and put $\mathfrak{h}_i' = \mathfrak{g}_0' \cap \mathfrak{h}_i$. Consider the mapping $\psi_i : (x^*, H) \to x^*H$ of $G_i^* \times \mathfrak{h}_i'$ into $\mathfrak{g}_0'$, and put $\psi_i(G_i^* \times \mathfrak{h}_i') = \mathfrak{g}_i$. Then ψ_i is everywhere regular on $G_i^* \times \mathfrak{h}_i'$, and therefore $\mathfrak{g}_i$ is open in $\mathfrak{g}_0$ and if w_i' is the order of W_i', any element x^*H $(x^* \in G_i^*, H \in \mathfrak{h}_i')$ of $\mathfrak{g}_i$ has exactly w_i' distinct pre-images under ψ_i, namely, $(x^*s, s^{-1}H_i)$ $(s \in W')$. Let d_iH denote the regular Euclidean measure on $\mathfrak{h}_i$. Then the invariant measure d_ix^* on G_i^* can be so normalized that

$$\int_{\mathfrak{g}_i} g(X)\,dX = \int_{G_i^* \times \mathfrak{h}_i'} |\pi_i(H)|^2 g(x^*H)\,d_ix^*d_iH \qquad (g \in C_c(\mathfrak{g}_i)).$$

Since no two among $\mathfrak{h}_1, \cdots, \mathfrak{h}_N$ are conjugate under G, it is clear that the sets $\mathfrak{g}_1, \cdots, \mathfrak{g}_N$ are disjoint. Moreover, if X is any regular element in $\mathfrak{g}_0$, its centralizer $\mathfrak{h}_X$ (in $\mathfrak{g}_0$) in a Cartan subalgebra of $\mathfrak{g}_0$ which must be conjugate to some $\mathfrak{h}_i$. This proves that $\mathfrak{g}_0' = \bigcup_{1 \leqq i \leqq N} \mathfrak{g}_i$, and therefore

$$\int_{\mathfrak{g}_0} g(X)\,dX = \int_{\mathfrak{g}_0'} g(X)\,dX = \sum_{1 \leqq i \leqq N} \int_{G_i^* \times \mathfrak{h}_i'} |\pi_i(H)|^2 g(x^*H)\,d_ix^*d_iH \qquad (g \in C_c(\mathfrak{g}_0)).$$

Now for any $f \in \mathfrak{B}(\mathfrak{g}_0)$, put

$$\phi_f^{(i)}(H) = \pi_i(H) \int_{G_i^*} f(x^*H)\,d_ix^* \qquad (H \in \mathfrak{h}_i').$$

From Theorem 3, we know that this integral is convergent, and it follows from Fubini's Theorem that[18]

$$\int_{\mathfrak{g}_0} \Xi f\,dX = \sum_{1 \leqq i \leqq N} \int_{\mathfrak{h}_i'} (\operatorname{conj} \pi_i(H))\,\Xi(H)\,\phi_f^{(i)}(H)\,d_iH \qquad (f \in \mathfrak{B}(\mathfrak{g}_0)).$$

Now put $G_1^* = G^*$, $d_1x^* = dx^*$, $d_1H = dH$ and $\pi_1 = \pi$ as before and select two real-valued nonnegative functions α_0 and β_0 in $C_c^\infty(G^*)$ and $C_c^\infty(\mathfrak{h}_0')$, respectively, such that

$$\int_{G^*} \alpha_0(x^*)\,dx^* = \int_{\mathfrak{h}_0} |\pi(H)|^2 \beta_0(H)\,dH = 1.$$

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[17] All the facts stated here have already been seen in Sections 6 and 8.

[18] We denote the complex conjugate of c by conj c $(c \in C)$.

Put $\alpha(x^*) = (w_1')^{-1} \sum_{s \in W_1'} \alpha_0(x^*s)$ and [12] $\beta = w^{-1} \sum_{s \in W} \beta_0{}^s$, where $W = W_1$ and w is the order of W. We have seen in Section 8 that the measure dx^* is invariant under the transformation $x^* \to x^*s$ of G^* for any fixed $s \in W_1'$. Hence

$$\int_{G^*} \alpha(x^*)\,dx^* = \int_{\mathfrak{h}_0} |\pi(H)|^2 \beta(H)\,dH = 1,$$

and it is obvious that there exists a function $F \in C_c^\infty(\mathfrak{g}_1)$ such that $F(x^*H) = \alpha(x^*)\beta(H)$ $(x^* \in G^*, H \in \mathfrak{h}_0')$. Consider its Fourier transform $\tilde{F}$ and, for any fixed i $(1 \leqq i \leqq N)$, put $\mathfrak{h}_i = \bar{\mathfrak{h}}_0$. Then since

$$\begin{aligned}\tilde{F}(X') &= \int \exp\{(-1)^{\frac{1}{2}} B(X', X)\} F(X)\,dX \\ &= \int_{G^* \times \mathfrak{h}_0'} \exp\{(-1)^{\frac{1}{2}} B(X', x^*H)\} \alpha(x^*) |\pi(H)|^2 \beta(H)\,dx^*\,dH,\end{aligned}$$

it is obvious from Lemma 37 that

$$\phi_{\tilde{F}}{}^{(i)}(\bar{H}) = (-1)^r \tau_{\bar{H}}(\pi\beta) \qquad (\bar{H} \in \bar{\mathfrak{h}}_0')$$

in the notation of Section 9. Moreover $(\pi\beta)^s = \epsilon(s)\pi\beta$ $(s \in W)$, and so it follows from Lemma 39 that

$$\tau_{\bar{H}}(\pi\beta) = w^{-1}(-1)^r \Phi_{\bar{H}}(\pi\beta).$$

Therefore if $i \neq 1$, $\phi_{\tilde{F}}{}^{(i)} = 0$ from Corollary 2 to Lemma 39. Now suppose $i = 1$. Then if we write $\phi_{\tilde{F}}$ instead of $\phi_{\tilde{F}}{}^{(1)}$, we find from the Corollary to Lemma 40 that

$$\sum_{s \in W} \epsilon(s) \phi_{\tilde{F}}(sH') = c_0 \int_{\mathfrak{h}_0} \exp\{(-1)^{\frac{1}{2}} B(H', H)\} \pi(H) \beta(H)\,dH \qquad (H' \in \mathfrak{h}_0')$$

since $(\pi\beta)^s = \epsilon(s)\pi\beta$ $(s \in W)$. On the other hand, $\tilde{F} \in \mathcal{B}(\mathfrak{g}_0)$, and therefore

$$\tilde{F}(0) = \int_{\mathfrak{g}_0} \Xi \partial(\omega^m) \tilde{F}\,dX = \sum_{1 \leqq i \leqq N} \int_{\mathfrak{h}_0'} (\mathrm{conj}\,\pi_i(H)) \phi_{F_m}{}^{(i)}(H) \Xi(H)\,dH,$$

where $F_m = \partial(\omega^m)\tilde{F}$. But from Theorem 3,

$$\phi_{F_m}{}^{(i)} = \partial(\omega_i{}^m) \phi_{\tilde{F}}{}^{(i)}$$

where ω_i is the restriction of ω on $\mathfrak{h}_i$. Therefore $\phi_{F_m}{}^{(i)} = 0$ if $i \neq 1$, and hence

$$\tilde{F}(0) = (-1)^r \int_{\mathfrak{h}_0'} \pi(H) \phi_{\tilde{F}}(H; \partial(\bar{\omega}^m)) \Xi(H)\,dH,$$

where $\bar{\omega} = \omega_1$. But since $\Xi(H) = \xi(\omega(H))$ $(H \in \mathfrak{h}_0')$ and $\bar{\omega}^s = \bar{\omega}$ $(s \in W)$, it is obvious that $\Xi(sH) = \Xi(H)$. Moreover, $\pi^s = \epsilon(s)\pi$, and so we conclude that

$$\tilde{F}(0) = (-1)^r w^{-1} \int_{\mathfrak{h}_0'} \pi(H)\phi'(H; \partial(\bar{\omega}^m))\Xi(H)\,dH,$$

where $\phi'(H) = \sum_{s \in W} \epsilon(s)\phi_{\tilde{F}}(sH)$ $(H \in \mathfrak{h}_0')$. On the other hand,

$$\begin{aligned}\tilde{F}(0) = \int_{\mathfrak{g}_0} F(X)\,dX &= \int_{G^* \times \mathfrak{h}_0'} F(x^*H)\,|\pi(H)|^2\,dx^*dH \\ &= \int \alpha(x^*)\,dx^* \int \beta(H)\,|\pi(H)|^2\,dH = 1.\end{aligned}$$

Therefore ϕ' cannot be identically zero on $\mathfrak{h}_0'$, and this proves that $c_0 \neq 0$.

LEMMA 41. *Suppose $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ which is contained in $\mathfrak{k}_0$. Let $\mathfrak{h}_{(i)}$ $(1 \leqq i \leqq q)$ be all the distinct connected components of $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{g}_0'$. Then there exists a function $g \in \mathcal{C}(\mathfrak{g}_0)$ such that*

$$\sum_{1 \leqq i \leqq q} \operatorname*{Lim}_{\mathfrak{h}_{(i)}} \phi_g(0; \partial(\pi)) \neq 0$$

in the notation of Section 7. Moreover the constant c_0 of Theorem 4 is not zero.

Put $g = \tilde{F}$. Then since $\pi^s = \epsilon(s)\pi$ $(s \in W)$, it is clear that

$$\begin{aligned}\phi'(H'; \partial(\pi)) &= \sum_{s \in W} \phi_g(sH'; \partial(\pi)) \\ &= c_0(-1)^{r/2} \int_{\mathfrak{h}_0} \exp\{(-1)^{\frac{1}{2}} B(H', H)\}\pi(H)^2\beta(H)\,dH \\ &\qquad (H' \in \mathfrak{h}_0'),\end{aligned}$$

and therefore

$$\operatorname*{Lim}_{H' \to 0} \sum_{s \in W} \phi_g(sH'; \partial(\pi)) = c_0(-1)^{r/2} \int_{\mathfrak{h}_0} \pi(H)^2\beta(H)\,dH = c_0(-1)^{r/2} \neq 0.$$

Since every $s \in W$ permutes $\mathfrak{h}_{(1)}, \cdots, \mathfrak{h}_{(q)}$ among themselves, the statement of the lemma is now obvious.

12. Appendix.

LEMMA 42. *Let U be an open convex subset of a real Euclidean space E of dimension l, and let $\sigma_1, \cdots, \sigma_N$ be a finite number of vector subspaces of E of dimension $\leqq l-2$. Let U' denote the complement in U of the set $U \cap (\bigcup_{1 \leqq i \leqq N} \sigma_i)$. Then if X_1, X_2 are any two points in U', we can select*

$X_2' \in U'$ arbitrarily near X_2 such that the straight line-segment joining X_1 and X_2' lies entirely in U'.

By replacing σ_i by a larger linear subspace, if necessary, we may clearly assume that $\dim \sigma_i = l-2$ $(1 \leqq i \leqq N)$. For each i, select two linear functions α_i, β_i on E such that σ_i is exactly the set of those $X \in E$ where $\alpha_i(X) = \beta_i(X) = 0$. Obviously, α_i and β_i are linearly independent for every i. Let E' denote the complement of $\bigcup_{1 \leqq i \leqq N} \sigma_i$ in E. If Y_1, Y_2 are two points in E', it is evident that the line-segment joining Y_1, Y_2 cannot meet σ_i unless $\{\alpha_i(Y_1)/\beta_i(Y_1)\} = \{\alpha_i(Y_2)/\beta_i(Y_2)\}$. Now suppose X_1, X_2 are two points in U' and Σ is the set of those i $(1 \leqq i \leqq N)$ for which $\{\alpha_i(X_1)/\beta_i(X_1)\} = \{\alpha_i(X_2)/\beta_i(X_2)\}$. For the proof of the lemma, we may obviously assume that Σ is not empty. Then since α_j, β_j are linearly independent for every j, we can select $X \in E$ such that

$$\alpha_i(X)\beta_i(X_1) - \beta_i(X)\alpha_i(X_1) \neq 0$$

for all $i \in \Sigma$. Then if ϵ is a sufficiently small positive number and $X_2' = X_2 + \epsilon X$, it is clear that $X_2' \in U'$ and

$$\alpha_j(X_2')\beta_j(X_1) - \beta_j(X_2')\alpha_j(X_1) \neq 0 \qquad (1 \leqq j \leqq N).$$

This proves that the line-segment joining X_1 and X_2' lies in $U \cap E' = U'$.

COROLLARY. *U' is connected.*

Now we use the notation of Section 6. Define $\mathfrak{h}_0'$ and $\mathfrak{h}_1$ as in Lemma 24 and let Q be the set of those roots $\alpha \in P$ for which α^2 takes only real values on $\mathfrak{h}_0$. Since all roots take only real values on $(-1)^{\frac{1}{2}}\mathfrak{h}_{t_0} + \mathfrak{h}_{p_0}$, it is obvious that a positive root lies in Q if and only if it vanishes identically either on $\mathfrak{h}_{t_0}$ or on $\mathfrak{h}_{p_0}$. Let P' denote the complement of Q in P, and, for any $\alpha \in P$, let σ_α denote the set of those $H \in \mathfrak{h}_0$ where $\alpha(H) = 0$. Then σ_α is a linear subspace of $\mathfrak{h}_0$, and if $l = \dim_R \mathfrak{h}_0$, it is clear that $\dim_R \sigma_\alpha = l-1$ or $l-2$ according as α lies in Q or P'.

For any $\alpha \in Q$, define $\alpha^* = \alpha$ or $(-1)^{\frac{1}{2}}\alpha$ according as α takes only real or only pure imaginary values on $\mathfrak{h}_0$. Then since $\mathfrak{h}_1$ is connected, α^* must keep constant sign on $\mathfrak{h}_1$. Put $\alpha' = \epsilon_\alpha \alpha^*$, where $\epsilon_\alpha = 1$ or -1 according as α^* is positive or negative on $\mathfrak{h}_1$. Let $\mathfrak{h}_2$ be the set of all $H \in \mathfrak{h}_0$ where $\alpha'(H) > 0$ $(\alpha \in Q)$. Clearly, $\mathfrak{h}_2$ is an open convex subset of $\mathfrak{h}_0$ and $\mathfrak{h}_2 \cap \mathfrak{h}_0'$ is the complement in $\mathfrak{h}_2$ of the set $\mathfrak{h}_2 \cap (\bigcup_{\alpha \in P'} \sigma_\alpha)$. Therefore it follows from

the Corollary of Lemma 42 that $\mathfrak{h}_2 \cap \mathfrak{h}_0'$ is connected. On the other hand, it is obvious that $\mathfrak{h}_2 \cap \mathfrak{h}_0' \supset \mathfrak{h}_1$. Therefore, since $\mathfrak{h}_1$ is a maximal connected subset of $\mathfrak{h}_0'$, we conclude that $\mathfrak{h}_1 = \mathfrak{h}_2 \cap \mathfrak{h}_0'$. So the following result is an immediate consequence of Lemma 42.

LEMMA 43. *If H_1, H_2 are any two points in $\mathfrak{h}_1$, we can select $H_2' \in \mathfrak{h}_1$ arbitrarily near H_2 such that the straight line-segment joining H_1 and H_2' lies entirely in $\mathfrak{h}_1$.*

Remark. Let V_1 be the closure of $\mathfrak{h}_1$ in $\mathfrak{h}_0$. Since $\mathfrak{h}_1 = \mathfrak{h}_2 \cap \mathfrak{h}_0'$ is dense in $\mathfrak{h}_2$, it is obvious that V_1 is the set of those points $H \in \mathfrak{h}_0$ where $\alpha'(H) \geqq 0$ $(\alpha \in Q)$. Moreover, $V_1 \cap \mathfrak{h}_0' = \mathfrak{h}_2 \cap \mathfrak{h}_0' = \mathfrak{h}_1$. We shall need these facts a little later.

For a linear function λ on $\mathfrak{h}$ and $s \in W$, define the linear function $s\lambda$ by $(s\lambda)(H) = \lambda(s^{-1}H)$ $(H \in \mathfrak{h})$. Then if α is a root, the same holds for $s\alpha$. Let W_R be the subgroup consisting of those elements in W which map $\mathfrak{h}_0$ onto itself. Then if α^2 is real on $\mathfrak{h}_0$, the same obviously holds for $(s\alpha)^2$ $(s \in W_R)$. This means that s permutes $\{\sigma_\alpha\}_{\alpha \in Q}$ and $\{\sigma_\alpha\}_{\alpha \in P'}$ separately among themselves. Let s_α denote the Weyl reflexion corresponding to any root α. Then $s_\alpha H = H - 2\{\alpha(H)/\alpha(H_\alpha)\}H_\alpha$ $(H \in \mathfrak{h})$, where H_α is the element in $\mathfrak{h}$ determined by the condition $B(H, H_\alpha) = \alpha(H)$ $(H \in \mathfrak{h})$. If $\alpha \in Q$, it is clear that $\alpha(H)H_\alpha \in \mathfrak{h}_0$ for $H \in \mathfrak{h}_0$, and therefore $s_\alpha \in W_R$. Let W_Q be the subgroup of W generated by s_α $(\alpha \in Q)$. Then $W_Q \subset W_R$.

LEMMA 44. $\mathfrak{h}_0' = \bigcup_{s \in W_Q} s\mathfrak{h}_1$.

Let U be the complement in $\mathfrak{h}_0$ of the union of $\sigma_\alpha \cap \sigma_\beta$ $(\alpha, \beta \in P, \alpha \neq \beta)$. Since every root takes only real values on $\mathfrak{h}_{p_0} + (-1)^{\frac{1}{2}}\mathfrak{h}_{t_0}$, it is obvious that[18] $\alpha(\theta(H)) = -\operatorname{conj}\alpha(H)$ $(H \in \mathfrak{h}_0, \alpha \in P)$. Therefore[19] $\alpha = \pm\theta\alpha$ if and only if $\alpha \in Q$. Hence it follows from the definition of U that no root in P' can vanish anywhere on U. Moreover, it is obvious that $sU = U$ $(s \in W_R)$, and from the Corollary to Lemma 42, U is connected. Let V_1 denote the closure of $\mathfrak{h}_1$ in $\mathfrak{h}_0$, and put $V = \bigcup_{s \in W_Q} sV_1$. We shall first prove that $V = \mathfrak{h}_0$.

Since U is connected and dense in $\mathfrak{h}_0$, it would be enough to show that $V \cap U$ is open. But $V \cap U = \bigcup_{s \in W_Q} s(V_1 \cap U)$. Therefore it would be sufficient to prove that any point $H_0 \in V_1 \cap U$, lies in the interior of $V \cap U$.

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[19] For any linear function λ on $\mathfrak{h}$, $\theta\lambda$ denotes the function $H \to \lambda(\theta(H))$ $(H \in \mathfrak{h})$.

Since $\mathfrak{h}_1 \subset V \cap U$ and $\mathfrak{h}_1$ is open, we may suppose that $H_0 \notin \mathfrak{h}_1$. Then, in view of the definition of U, there is exactly one root $\alpha_0 \in P$ such that $H_0 \in \sigma_{\alpha_0}$ and moreover $\alpha_0 \in Q$. Therefore $\dim \sigma_{\alpha_0} = l-1$ and s_{α_0} is the reflexion in the hyperplane σ_{α_0}. Select a positive number ϵ, and let N be the set of those points $H \in \mathfrak{h}_0$ where $\alpha_0'(H) \geqq 0$ and $\| H - H_0 \| \leqq \epsilon$. Then if ϵ is sufficiently small, it is clear that $\alpha'(H) \geqq \frac{1}{2}\alpha'(H_0)$ for any $H \in N$ and α in Q. Therefore $N \subset V_1$ (see the remark after Lemma 43). But it is obvious that $N \cup s_{\alpha_0} N$ consists of all points $H \in \mathfrak{h}_0$ for which $\| H - H_0 \| \leqq \epsilon$, and therefore, since U is open, H_0 lies in the interior of $V \quad U$. This proves that $V = \mathfrak{h}_0$.

Now we have seen above that $V_1 \cap \mathfrak{h}_0' = \mathfrak{h}_1$. Therefore

$$\mathfrak{h}_0' = V \quad \mathfrak{h}_0' = \bigcup_{s \in W_Q} s(V_1 \cap \mathfrak{h}_0') = \bigcup_{s \in W_Q} s\mathfrak{h}_1.$$

Thus the lemma is proved.

Let Q_+ be the set of those roots $\alpha \in Q$ which take only real values on $\mathfrak{h}_0$. It is obvious that a positive root β lies in Q_+ if and only if $H_\beta \in \mathfrak{h}_{\mathfrak{p}_0}$. Hence $Q_+ \subset P_+$ and $Q = Q_+ \cup P_0 \cup P_-$ in the notation of Section 5.

LEMMA 45. *If Q_+ is not empty, there exists a root $\alpha \in Q_+$ such that $V_1 \quad U \cap \sigma_\alpha$ is not empty.*

For otherwise suppose $V_1 \cap U \cap \sigma_\alpha = \emptyset$ for every $\alpha \in Q_+$. Let W_1 be the subgroup of W_Q generated by s_β for $\beta \in P_0 \cup P_-$. Then if we use the notation of the proof of Lemma 44, it is clear that $H_0 \in V_1 \cap U \cap \sigma_{\alpha_0}$ and therefore $\alpha_0 \in P_0 \cup P_-$. Hence it follows immediately that $\bigcup_{s \in W_1} s(V_1 \cap U)$ is open, and therefore $\mathfrak{h}_0 = \bigcup_{s \in W_1} sV_1$ and $\mathfrak{h}_0' = \bigcup_{s \in W_1} s\mathfrak{h}_1$. Now select a root $\alpha \in Q_+$. Then α keeps constant sign on $\mathfrak{h}_1$. Moreover since $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{h}_\mathfrak{k}$ are orthogonal, it is obvious that $s\alpha = \alpha$ for all $s \in W_1$. Hence α keeps constant sign on $\mathfrak{h}_0' = \bigcup_{s \in W_1} s\mathfrak{h}_1$. As this is evidently false, our assertion follows.

LEMMA 46. *For any $\alpha \in P$, we can select X_α, $X_{-\alpha}$ in such a way that $\tilde{\theta}(X_\alpha) = -X_{-\alpha}$ and $[X_\alpha, X_{-\alpha}] = \{2/\alpha(H_\alpha)\}H_\alpha$. Moreover, if $\alpha \in Q_+$, we can assume that $X_\alpha, X_{-\alpha} \in \mathfrak{g}_0$.*

Since $\operatorname{conj} \alpha(H) = -\alpha(\theta(H)) \quad (H \in \mathfrak{h}_0)$, it follows that $[H, \tilde{\theta}(X_\alpha)] = -\alpha(H)\tilde{\theta}(X_\alpha) \quad (H \in \mathfrak{h}_0)$, and therefore we can assume that $X_{-\alpha} = -\tilde{\theta}(X_\alpha)$.

Then $B(X_\alpha, X_{-\alpha}) = -B(X_\alpha, \tilde{\theta}(X_\alpha)) = \| X_\alpha \|^2$. Now normalize X_α in such a way that $\| X_\alpha \|^2 = 2/\alpha(H_\alpha)$. Then if $H \in \mathfrak{h}$,

$$B([X_\alpha, X_{-\alpha}], H) = B(X_{-\alpha}, [H, X_\alpha]) = \alpha(H)B(X_{-\alpha}, X_\alpha) = \{2/\alpha(H_\alpha)\}\alpha(H),$$

and this shows that $[X_\alpha, X_{-\alpha}] = \{2/\alpha(H_\alpha)\}H_\alpha$. Now suppose $\alpha \in Q_+$. Then if ζ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, it is clear that $[H, \zeta(X_\alpha)] = \alpha(H)\zeta(X_\alpha)$ $(H \in \mathfrak{h}_0)$. Moreover $\| \zeta(X) \| = \| X \|$ for any $X \in \mathfrak{g}$. Hence $\zeta(X_\alpha) = c^2 X_\alpha$, where c is a unimodular complex number. Therefore, replacing X_α by cX_α, we can assume that $\zeta(X_\alpha) = X_\alpha$ without disturbing the value of $\| X_\alpha \|$. Then $X_\alpha \in \mathfrak{g}_0$ and $X_{-\alpha} = -\tilde{\theta}(X_\alpha) = -\theta(X_\alpha)$ is also in $\mathfrak{g}_0$.

COROLLARY 1. *For a fixed $\alpha \in P$, select $X_\alpha, X_{-\alpha}$ according to the above lemma and put $\mathfrak{l} = CH_\alpha + CX_\alpha + CX_{-\alpha}$. Then*

$$\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0 = \begin{cases} RH_\alpha + RX_\alpha + RX_{-\alpha} \text{ if } \alpha \in Q_+, \\ R(-1)^{\frac{1}{2}}H_\alpha + R(-1)^{\frac{1}{2}}(X_\alpha - X_{-\alpha}) + R(X_\alpha + X_{-\alpha}) \text{ if } \alpha \in P_0, \\ R(-1)^{\frac{1}{2}}H_\alpha + R(X_\alpha - X_{-\alpha}) + R(-1)^{\frac{1}{2}}(X_\alpha + X_{-\alpha}) \text{ if } \alpha \in P_-. \end{cases}$$

If $\alpha \in Q_+$, our assertion is obvious. Now suppose $\alpha \in P_0$. Then $X_\alpha \in \mathfrak{p}$, and therefore

$$X_{-\alpha} = -\tilde{\theta}(X_\alpha) = \zeta(X_\alpha).$$

Therefore $X_\alpha + X_{-\alpha}$ and $(-1)^{\frac{1}{2}}(X_\alpha - X_{-\alpha})$ are in $\mathfrak{l}_0$ in this case. Similarly, $X_\alpha \in \mathfrak{k}$ and $X_{-\alpha} = -\tilde{\theta}(X_\alpha) = -\zeta(X_\alpha)$ if $\alpha \in P_-$. Therefore $X_\alpha - X_{-\alpha}$ and $(-1)^{\frac{1}{2}}(X_\alpha + X_{-\alpha})$ are in $\mathfrak{l}_0$ in this case. Moreover, it is obvious that $\dim_R \mathfrak{l}_0 \leqq \dim_C \mathfrak{l} \leqq 3$, and so the above statements follow.

Put $H' = 2\alpha(H_\alpha)^{-1}H_\alpha$, $X' = X_\alpha$, $Y' = X_{-\alpha}$. Then $[H', X'] = 2X'$, $[H', Y'] = -2Y'$, $[X', Y'] = H'$, and a simple calculation shows that $\exp(\frac{\pi}{2}\operatorname{ad}(X' - Y'))H' = -H'$. On the other hand, it is obvious that $\exp(\frac{\pi}{2}\operatorname{ad}(X' - Y'))H = H$ if $\alpha(H) = 0$ $(H \in \mathfrak{h})$. Therefore

$$\exp(\tfrac{\pi}{2}\operatorname{ad}(X' - Y'))H = s_\alpha H \qquad (H \in \mathfrak{h}).$$

COROLLARY 2. *If $\alpha \in Q_+ \cup P_-$ there exists an element $x \in K$ such that $xH = s_\alpha H$ for all $H \in \mathfrak{h}$.*

This follows from the fact that $X' - Y' \in \mathfrak{k}_0$ if $\alpha \in Q_+ \cup P_-$.

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RESTRICTED BASES.*

By JOHN B. KELLY.

1. Introduction. A set of non-negative integers is called a basis of order h if every positive integer can be expressed as a sum of h elements of the set and if h is the smallest number for which this is true. A set of non-negative integers is an asymptotic basis of order h if every sufficiently large integer can be expressed as a sum of h or fewer elements of the set and if h is the smallest number for which this is true. Finally, we shall call a set of non-negative integers a restricted basis of order k if every sufficiently large positive integer can be expressed as a sum of k or fewer elements of the set *without repetitions* and if k is the smallest number for which this is true.

The set of squares is, according to Lagrange's theorem, a basis of order 4, and, according to Pall's theorem [1], a restricted basis of order 5. One may conjecture that this example is typical of the general situation; i.e., that every basis or every asymptotic basis is a restricted basis. However, a simple counter-example due to P. T. Bateman shows that this is not the case: If $h > 2$, the set consisting of 1 and all non-negative multiples of h is a basis of order h but not a restricted basis of any order. Evidently some kind of divisibility condition must be imposed upon a basis to force it to be a restricted basis.

In this paper we discuss the case $h = 2$, for which the above counter-example is invalid. We shall prove the following theorems:

THEOREM I. *A basis of order 2 is a restricted basis of order at most 4.*

THEOREM II. *An asymptotic basis of order 2 is a restricted basis of order at most 3 if the value of its counting function is, for a suitable positive constant C, larger than $Cx/\log\log x$ for all sufficiently large x.*

(The value of the counting function of a set of integers is the number of integers in the set $\leqq x$; here x is a non-negative real number.)

It is possible that every basis, or even every asymptotic basis, of order 2 is a restricted basis of order at most 3, but we have been unable to establish

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this. As another simple example shows, this result, if true, is best possible: The set of integers whose representations in the ternary scale have only the digits 0 and 1 is a basis of order 2 but not a restricted basis of order 2.

Our proof of Theorem I is elementary. It involves certain lemmas which bear somewhat upon a famous conjecture of Erdös and Turán [2]. Apart from the use that is made of Roth's celebrated theorem on progression-free sequences [3], our proof of Theorem II is also elementary.

2. The main lemmas. Let A be a basis of order 2 with elements a_i, $i=1,2,3,\cdots$. By a' we shall mean the smallest element in A which is larger than a. If x is an arbitrary integer, we shall mean by $\rho(x)$ the number of representations of x as the sum of two elements of A, where the representations $x=a_i+a_j$ and $x=a_j+a_i$ will be regarded as distinct when $i\neq j$.

Theorem I is an easy consequence of the following lemmas:

LEMMA 1. *If A is a basis of order 2, there exists an even integer x such that $\rho(x)\geqq 4$.*

LEMMA 2. *If A is a basis of order 2, then either there exists an odd integer y such that $\rho(y)\geqq 4$ or A is one of the two sets*

$$(0,1,3,\cdots,2n+1,\cdots), \quad (0,1,2,4,\cdots,2^n,\cdots).$$

In 1941, Erdös and Turán conjectured that $\limsup \rho(n)=\infty$ for asymptotic bases. Either Lemma 1 or Lemma 2 implies that, for bases, l. u. b. $\rho(n)\geqq 4$. Actually, this can be shown directly, and more simply, using the method employed in proving Lemmas 1 and 2.

3. Proof of Lemma 1. We shall show, in order to prove Lemma 1, that the assumption that every even number has no more than 3 representations as a sum of two integers of a basis, A, of order 2, leads to a contradiction. Our method involves a cumbersome consideration of various cases. Clearly $0\in A$, $1\in A$ and either 2 or 3 or both are in A.

Case 1. $2\in A$, $3\in A$. Here $3'\neq 4$, for otherwise $\rho(4)>3$. If $3'=5$, then $5'=9$, for $5'<9\rightarrow\rho(8)>3$ while $5'>9\rightarrow\rho(9)=0$. Similarly $9'=13$. But now $\rho(14)=4$; hence $3'\neq 5$. If $3'=6$, then $6'\geqq 9$, for $6'<9\rightarrow\rho(8)>3$. If $6'=9$, one sees, arguing in the same way, that the

next elements in A must be 13, 17, and 21; but then $\rho(30) > 3$. Therefore $6' = 10$, for otherwise $\rho(10) = 0$. As before, the next elements in A must be 17, 21, 25 and 29; but then $\rho(46) > 3$. Thus $3' \neq 6$. Consequently $3' = 7$, for otherwise $\rho(7) = 0$. But now $7' = 11$ and $11' = 15$ and we have $\rho(18) > 3$. We have shown that A cannot contain both 2 and 3.

Case 2. $2 \in A$, $3 \notin A$. If $2' = 4$, then $4' = 7$ and $7' \geqq 9$. If $7' = 9$, then 10, 12, 14, 15 and 16 are not in A, for then $\rho(16) > 3$ or $\rho(10) > 3$. Thus $9' = 11$, since otherwise $\rho(12) = 0$. $11' \neq 13$, for then $\rho(20) > 3$. Hence $11' = 17$, since otherwise $\rho(17) = 0$. But now $\rho(18) > 3$. Hence $7' \neq 9$. Thus $7' = 10$, otherwise $\rho(10) = 0$. 11 and 12 are not in A, for then $\rho(12) > 3$. $13 \notin A$, for then $\rho(14) > 3$. But now $\rho(13) = 0$. Hence $2' \neq 4$.

It follows that $2' = 5$, for otherwise $\rho(5) = 0$. Now $5' \neq 6$, for then $\rho(6) > 3$. If $5' = 7$, then $7' \geqq 9$. If $7' = 9$, then 10, 11, and 12 are not in A, for then $\rho(12) > 3$. Also, $13 \notin A$, for then $\rho(14) > 3$. But now $\rho(13) = 0$. Hence $7' \neq 9$. If $7' = 10$ or $7' = 11$, then $\rho(12) > 3$. Hence $7' > 11$. But now $\rho(11) = 0$. Thus $5' \neq 7$. Consequently $5' = 8$, for otherwise $\rho(8) = 0.8' \geqq 11$, for otherwise $\rho(10) > 3$. But if $8' > 11$, then $\rho(11) = 0$; hence $8' = 11$. $11' \geqq 13$, for otherwise $\rho(12) > 3$. If $11' = 13$, then one can see that 14, 15, 16 and 17 are not in A, for the presence of any one of these would make $\rho(14) > 3$ or $\rho(16) > 3$ or $\rho(18) > 3$. But now $\rho(17) = 0$. Hence $11' = 14$, for otherwise $\rho(14) = 0$. But now $\rho(16) > 3$. Thus the case $2 \in A$, $3 \notin A$ cannot occur.

Case 3. $2 \notin A$, $3 \in A$. $3' \neq 4$; otherwise $\rho(4) > 3$. Thus $3' = 5$, as otherwise $\rho(5) = 0$. $5' \neq 6$, for then $\rho(6) > 3$. Hence $5' = 7$, as otherwise $\rho(7) = 0$. But now $\rho(8) > 3$. This completes the proof of Lemma 1.

4. Proof of Lemma 2. Let A be a basis of order 2 and suppose that every odd number has exactly 2 representations as the sum of 2 elements of A. As before, $0 \in A$ and $1 \in A$.

Case 1. $2 \in A$. We show that there are no larger odd numbers in A. Suppose the contrary. Let $2m + 1$, $m \geqq 1$, be the smallest odd number, apart from 1, in A. Then A must contain all even numbers between 2 and $2m - 2$ inclusive, for otherwise the odd numbers between 3 and $2m - 1$ would have no representations. It is also clear that $2m \notin A$. Since every odd number $\leqq 4m - 1$ now has two representations, it follows that

$(2m+1)' > 4m-1$. But then $\rho(4m-2) = 0$, unless $m = 1$ or $m = 2$. The case $m = 1$ is excluded immediately, since here $\rho(3) > 2$. In the latter case, $2' = 5$, and $5' = 8$. Clearly $9 \notin A$; also $11 \notin A$, as otherwise $\rho(13) > 2$. Hence $8' = 10$, for otherwise $\rho(11) = 0$. 12 and 13 are not in A, for then $\rho(13) > 2$. $14 \notin A$, for then $\rho(15) > 2$. But now $\rho(14) = 0$.

Thus the only odd element of A in this case is 1. It follows that all even numbers must be in A; that is, A is the set $(0, 1, 2, \cdots, 2n, \cdots)$.

Case 2. $2 \notin A$. We show that all remaining elements of A are odd. Suppose the contrary. Let $2m$, $m \geqq 2$, be the smallest even element of A, apart from 0. Then A must contain all odd numbers $\leqq 2m-1$. It follows, as before, that $(2m)' = 4m$ or $(2m)' = 4m+1$. If $(2m)' = 4m$, then $(4m)' > 4m+2$, as otherwise $\rho(4m+1) > 2$ or $\rho(4m+3) > 2$. But now $\rho(4m+2) = 0$.

Hence $(2m)' = 4m+1$. By the same arguments as before we infer that A contains all odd numbers in the closed interval $[4m+1, 6m-1]$ and no even numbers in this interval. All odd numbers less than $8m-1$ are now represented as sums of elements of A; it follows that there are no integers in A in the closed interval, $[6m, 8m-1]$. $8m \in A$; otherwise $\rho(8m) = 0$. All odd numbers up to $10m-1$ now having 2 representations, there are no elements of A between $8m$ and $10m-1$. Either $(8m)' = 10m$ or $(8m)' = 10m+1$; in the contrary case $\rho(10m+1) = 0$.

If $(8m)' = 10m$, then all odd numbers $\leqq 14m-1$ have two representations; $(10m)' \geqq 14m$. But now $\rho(12m+2) = 0$. Hence $(8m)' = 10m+1$. But now $\rho(12m+1) > 2$.

Thus all elements of A, apart from 0, are odd; A is the set $(0, 1, 3, 5, \cdots)$.

5. Proof of Theorem I. Let A be a basis of order 2 and let z be an arbitrary odd positive integer larger than $3x$, where x is the even integer mentioned in Lemma 1. Then $z-x$ is odd and we may write $z-x = a_1 + a_2$, since A is a basis of order 2, with $a_1 \neq a_2$. Since $\rho(x) \geqq 4$, we have $x = a_3 + a_4 = a_5 + a_6$, where no two of the integers a_3, a_4, a_5, a_6 are the same. Since $z-x$ is larger than $2x$, at most one of the integers a_1, a_2 is less than or equal to x. Hence at most one of them can be equal to one of the integers a_3, a_4, a_5, a_6. Suppose, for example, that $a_1 = a_3$. Then we have $z = a_1 + a_2 + a_5 + a_6$ with all summands distinct.

In a similar fashion, using Lemma 2, we can treat the case in which

z is an arbitrary even integer larger than $3y$. Theorem I is obviously true for the exceptional bases mentioned in Lemma 2.

It has been pointed out by the referee that Theorem I is immediate if one knows that $\rho(n) < n^{1/3}/6^{2/3}$. For in this case we have for large n,

$$A^2(\tfrac{1}{2}n) < \sum_{j=0}^{n} \rho(n) < n^{4/3}/6^{2/3}, \text{ whence } A(\tfrac{1}{2}n) < n^{2/3}/6^{1/3}.$$

If $\lambda(n)$ denotes the number of representations of n as the sum of four elements of A with at least one repetition, then

$$\lambda(n) \leqq 6 \sum_{A} \rho(n-2j) \leqq 6A(\tfrac{1}{2}n)\rho(n) < n.$$

But, since A is a basis of order 2, the total number of representations of n as the sum of 4 elements of A is at least n.

6. Progressive-free sets. Let $B = (b_1, b_2, \cdots, b_r)$ be a set of distinct positive integers. B is said to be a progression-free set if the equation $2b_k = b_i + b_j$ has no solution with $i \neq j$. Let x be a positive real number and let $B(x)$ denote the maximum number of integers in any progression-free set of whose elements are less than x. Roth's theorem states that $B(x) = O(x/\log\log x)$.

Let S be any set of non-negative integers and let T be the set of integers which can be represented as the sum of 2 elements in S in just one way, namely as $2s_i$, $s_i \in S$. Then T is progression free, for $2(2s_i) = 2s_j + 2s_k$ implies $2s_i = s_j + s_k$, so that $i = j = k$.

If $B = (b_1, b_2, \cdots, b_r)$ is progression-free the set $C = (c_1, c_2, \cdots, c_r)$, where $c_i = \lambda b_i + \mu$, $\lambda \neq 0$, $i = 1, 2, \cdots, r$, also contains no arithmetic progressions.

7. Chains. We turn now to the proof of Theorem II. Let A be an asymptotic basis of order 2. Suppose that every integer larger than t can be represented as the sum of 2 elements of A. Let x be an integer greater than t which cannot be represented as the sum of 3 distinct elements of A. If $a_p \in A$, $a_p \leqq x$, then, since A is an asymptotic basis of order 2, either $x - a_p \leqq t$ or $x - a_p = 2a_q$ with $a_q \in A$ or $x - a_p = a_p + a_r$ with $a_r \in A$, whence $x - a_r = 2a_p$. By an x-chain we mean a collection of elements $a_1, a_2, \cdots, a_k$ in A such that

$$x - a_1 = 2a_2, x - a_2 = 2a_3, \cdots, x - a_{k-1} = 2a_k, \tag{1}$$

where the equations $x - a_e = 2a_1$ and $x - a_k = 2a_m$ with $a_e \in A$, $a_m \in A$ have no solutions. It is clear that, with at most t exceptions, every element in A which is $\leqq x$ belongs to one and only one x-chain.

Using the standard method for solving a linear difference equation, one can easily show that

$$a_n = \tfrac{2}{3}(x - 3a_1)(-\tfrac{1}{2})^n + \tfrac{1}{3}x; \qquad n = 1, 2, \cdots, k. \tag{2}$$

It follows from (2) that, except for the case $a_1 = x/3$ the elements of a chain are distinct. The number of distinct elements in a chain will be called its length and will be denoted by k.

Putting $n = k$ in (2) we see that since a_k is an integer, $x - 3a_1$ must be divisible by 2^{k-1}. A chain will be said to be associated with m if $x - 3a_1$ is exactly divisible by 2^m. Clearly, if $a_1 \neq x/3$, $k \leqq m + 1$; also $m \leqq u = [\log x/\log 2] + 1$. In case $a_1 = x/3$, we shall put $k = 1$, $m = 0$.

The integer $x - a_1$ has only the representation $x - a_1 = 2a_2$ as the sum of two elements of A, for $x - a_1 = a_1 + a_e$ would imply $x - a_e = 2a_1$ and a_1 would not be the initial element of its x-chain. From our remarks in (6) it follows that the initial elements of all x-chains form a progression-free set.

The initial elements of the x-chains associated with m also form a progression-free set. So do the integers of the form $(x - 3a_1)/2^m$ where a_1 runs over all the initial elements of chains associated with m. These integers must lie between $-x/2^{m-1}$ and $x/2^m$, that is, in an interval of length $3x/2^m$. If we let $\alpha(m)$ denote the number of x-chains associated with m then

$$\alpha(m) \leqq B(3x/2^m). \tag{3}$$

Similarly, the total number of x-chains is $\leqq B(x)$.

8. Proof of Theorem II. We have

$$A(x) \leqq t + \sum k \leqq t + \sum (m+1) \leqq t + \sum_{m=0}^{u} m\alpha(m) + \sum 1,$$

where the unindexed summations are taken over all x-chains. Putting $v = [\log x/2 \log 2]$ and employing (3) we have

$$A(x) \leqq t + \sum_{m=0}^{v} mB(3x/2^m) + \sum_{m=v+1}^{u} mB(3x/2^m) + B(x).$$

Using Roth's theorem, we have

$$\begin{aligned} A(x) &\leqq t + \sum_{m=0}^{v} 3c_1 mx/2^m \log\log(3x/2^m) + \sum_{m=v+1}^{u} mB(3x^{\frac{1}{2}}) + c_1 x/\log\log x \\ &\leqq t + \sum_{m=0}^{v} 3c_1 mx/2^m \log\log(3x^{\frac{1}{2}}) + 3x^{\frac{1}{2}} \sum_{m=v+1}^{u} m + c_1 x/\log\log x \\ &\leqq t + c_2 x/\log\log x \sum_{m=0}^{\infty} m/2^m + c_3 x^{\frac{1}{2}} \log^2 x + c_1 x/\log\log x \end{aligned}$$

$\leqq t + c_4 x/\log\log x$, where c_1, c_2, c_3 and c_4 are absolute constants. Since x was an integer which could not be represented as the sum of 3 distinct elements of A, it is clear that our last inequalty implies Theorem II.

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ORDINARY DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORMS.*

By AUREL WINTNER.

PART I.

On the case of integrability by Laplace transforms which corresponds to the Fuchsian case of a singular point of a linear differential equation.

1. Let $f(z)$ be regular on the half-plane $\mathrm{Re}\, z > 0$, and let $f(z)$ be representable as a Laplace-Stieltjes transform

$$f(z) = \int_0^\infty e^{-zt}\, d\alpha(t) \tag{1}$$

$(0 \leqq t < \infty)$ which is *absolutely* convergent for $\mathrm{Re}\, z > 0$. Then the function $f(z)$ will be called *of class* (A) (more specifically, of class (A_0), where the subscript refers to the abscissa of the boundary line), and its $\alpha = \alpha(t)$ will be denoted by $\alpha_f = \alpha_f(t)$. It is understood that $\alpha(t)$ is uniquely determined by $f(z)$ only after the normalization of an additive constant and of the values of $\alpha(t)$ at its discontinuity points (for instance, if $\alpha(0) = 0$ and $\alpha(t+0) = \alpha(t)$, where $0 \leqq t < \infty$); and that $\alpha_f(t)$ is of bounded variation on every finite interval $0 \leqq t \leqq T$ but need not stay bounded as $t \to \infty$, since (1) need not converge at $z = 0$. It is clear that

$$f(z) \to \alpha_f(+0) - \alpha_f(0) \quad \text{as} \quad \mathrm{Re}\, z \to \infty, \tag{1 bis}$$

and that $f(z)$ is bounded for $\mathrm{Re}\, z > \epsilon$ if $\epsilon > 0$.

Consider the solutions $w = w(z)$ of the differential equation

$$w'' + f(z)\, w = 0, \tag{2}$$

and let the trivial solution $w(z) \equiv 0$ (which is of class (A), with $\alpha_w(t) \equiv 0$) be excluded. Then, if $f(z)$ is of class (A), there need not exist any solution $w(z)$ of class (A) (or, for that matter, of class (A_r), no matter how large $r > 0$ be chosen, where $g(z)$ is called of class (A_r) if $g(z - r)$ is of class $(A_0) = (A)$). In order to see this, choose $\alpha_f(t) = \frac{1}{2}t^2$. Then (1) is of

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class (A) and reduces (2) to $w'' + w/z^2 = 0$. But this differential equation is satisfied (for $\mathrm{Re}\, z > 0$) by $w(z) = z^a$, where a is either root of $a(a-1)+1 = 0$; so that, since $\mathrm{Re}\, a = \frac{1}{2} > 0$ holds for both roots a, no solution $w(z)$ stays bounded as $Rz \to \infty$, and so no solution is of class (A) (or, for that matter, of class (A_r) for some $r > 0$).

It will turn out that this trivial counterexample is close to the border of that subclass of the functions $f(z)$ of class (A) for which the possibility of a counterexample becomes replaced by an existence statement. For, on one hand, such a subclass will prove to be characterized by the condition

$$\int_0^1 t^{-2} \,|\, da_f(t) \,|\, < \infty \tag{3}$$

(which is a local condition for $t \to 0$, since it is equivalent to

$$a_f(+0) = a_f(0) \tag{3 bis}$$

and $\int_\delta^\eta t^{-2} \,|\, da_f(t) \,|\, \to 0$ together, where $0 < \min(\delta, \eta) \to 0$). And, on the other hand, (3) is "just barely" (logarithmically) violated by the $a_f(t) = \frac{1}{2}t^2$ of the preceding counterexample $f(z)$.

2. It was shown in [7] that if $f(z)$, besides being of class (A), is such as to belong to an $a_f(t)$ which, for a sufficiently small $\epsilon > 0$, satisfies the condition

$$a_f(t) = \text{const. if } 0 \leqq t < \epsilon, \tag{4}$$

then (2) has a solution $w(z)$ of class (A), and that this $w(z)$ is substantially unique (by this is meant that this $w(z)$ can be replaced only by $cw(z)$, where c is just one of the two integration constants of (2)). But it was also shown in [7] that the restriction (4) of the coefficient function of (2) will compel the (A)-solution $w(z)$ of (2) to be such as to satisy $a_w(t) = \text{Const.}$ for $0 \leqq t < \epsilon$ (where ϵ is the same constant as in (4)). Correspondingly, the assumption (4) played the drastic part of completely eliminating "small divisors" (not "diophantine" small divisors, of the type occurring in celestial mechanics, but still "small divisors", entering in the discussion of the solution of the differential equation in the same way, and creating the same "resonance" trouble, as in celestial mechanics).

It was shown in [8] that if (2) is replaced by a system, say (S), of n homogeneous, linear differential equations of first order, with a coefficient

matrix $F(z)$ all n^2 elements of which are of class (A), then the drastic assumption corresponding to the case $f = F$ of (4) can be improved to an integral condition on the behavior of $a_F(t)$ near $t = 0$, a condition characterizing the allowable amount of "small divisors." But this problem on (S) is quite different from that of (2), since that integral condition on $a_F(t)$ refers to the systems (S) for which all (vector) solutions are of class (A), whereas (2), when written in the form (S) (with $n = 2$), can never have this property (correspondingly, $f(z) \equiv 1$ will be one of the $n^2 = 4$ of the elements of the resulting (S), and $f(z) \equiv 1$ belongs to $a_f(t) = \operatorname{sgn} t$ in (1); so that (3 bis) is violated). On the other hand, if an appropriate transcription of the method of [8] is attempted from (S) to (2), then the convergence of the successive approximations is made to depend on estimates which, though obtainable, become too involved.

In what follows, all these difficulties will be avoided by having recourse to function-theoretical arguments. Instead of applying, as in [7] and [8], successive approximations in the a-space, the successive approximations will be carried out *before* the application of the Laplace transformation. The necessary estimates then become of the classical type. But what is needed is their translation to the a-space. This can, however, be handled by an appeal to the theory of normal families, since the "translation" process is then taken care of by the "continuity theorem" of the transform (1) (where $f = f, w, w_k$; $k = 1, 2, \cdots$). Thus the proof becomes a counterpart (for Laplace transforms) of the corresponding treatment of the spectral resolution (which is an isue concerning Stieltjes transforms, instead of Laplace transforms); cf. [4], pp. 104-105 and pp. 171-173.

3. What will thus result is the following theorem:

If both coefficient functions $g(z)$, $f(z)$ of

$$w'' + g(z)w' + f(z)w = 0, \tag{5}$$

where $\operatorname{Re} z > 0$, are of class (A), and if their (Laplace-Stieltjes) Unterfunktionen $a_f(t)$, $a_g(t)$, where $0 \leqq t < \infty$, behave, near $t = 0$, so as to satisfy the respective "small divisor" restrictions

$$\int_0^1 t^{-1} \, |\, da_g(t)\,| < \infty, \qquad \int_0^1 t^{-2} \, |\, da_f(t)\,| < \infty, \tag{6}$$

then, on the half-plane $\operatorname{Re} z > 0$, the general solution of (5) is of the form

$$w(z) = c_1 z u(z) + c_2 v(z), \tag{7}$$

where the function $u(z)$ *(which is not a solution) and the function* $v(z)$ *(which is a solution* $w(z)$ *belonging to* $c_1 = 0$*) are of class* (A).

The structure of the two conditions (6), when compared with Fuchs' classical theorem on differential equations (5) for which $z = 0$ is a "regular" singular point (cf., e.g., [1], p. 120), explains the choice of the title of this Part I. In this regard, cf. also Bôcher's extension [2] of Fuchs' sufficient condition to the real field, particularly the case dealing with the case corresponding to a multiple elementary divisor, and the remark made in the footnote on p. 854 of [6].

Corresponding to the Fuchsian parallelism, it is clear how the assumption and the assertion of the theorems (and it will be obvious from the nature of the proof below how the proof of the theorem) can be extended from $n = 2$ to any n, if n is the order of the homogeneous, linear differential equation $D(w) = 0$. If $n = 1$, then the theorem is trivial; cf. [8], p. 174. This trivial case also shows that the "Fuchsian" integral conditions of the general theorem cannot (in general) be improved (in this regard, cf. also the Appendix of [5]). The theorem resulting for an arbitrary n is a "small divisor" refinement of the corresponding result of [3], where (4) is assumed for all n coefficient functions $f(z)$ of $D(w) = w^{(n)} + \cdots + f_0(z)w$.

In order to halve the length of the formulae, the proof of the italicized theorem will be given for the case $g(z) \equiv 0$ only. Then (5) and (6) reduce to (2) and (3) respectively. It will be clear from the proof that the modifications needed in it when $g(z) \not\equiv 0$ are of a trivial nature. Actually, the unessential assumption $g(z) \equiv 0$ will shorten the proof only insofar as it will make possible the use of some of the passages in [5] and [6].

4. Instead of the analytic differential equation (2), consider first the differential equation

$$w'' + f(x)w = 0 \tag{8}$$

($' = d/dx$) in which $f(x)$ is a (possibly complex-valued) continuous function of the real variable x, defined for large positive x in such a way that $xf(x)$ is absolutely integrable or, what is the same thing, that

$$F(x) \to 0 \text{ as } x \to \infty, \tag{9}$$

where

$$F(x) = \int_x^\infty s\,|f(s)|\,ds. \tag{10}$$

Under these assumptions on $f(x)$, the following facts are contained in Bôcher's paper [2] implicitly (cf. the remark on [2] made in Section 3 above), and in [5] explicitly (and, incidentally, under an assumption requiring of f somewhat less than the absolute integrability of xf):

(i) The differential equation (8) has a solution $w(x)$ satisfying the "initial condition"

$$w(x) \to 1 \text{ as } x \to \infty, \tag{11}$$

and this solution of (8) is unique.

(ii) On the half-line on which $f(x)$ is given, all the integrals occurring in the recursion formula

$$w_n(x) = 1 + \int_x^\infty (x-s) f(s) w_{n-1}(s)\, ds, \tag{12}$$

where $n = 1, 2, \cdots$ and

$$w_1(x) \equiv 0, \tag{13}$$

are absolutely convergent, and the differences of consecutive functions in the sequence

$$w_1(x), \cdots, w_n(x), \cdots \tag{14}$$

are subject, at every point x of that half-line, to the inequality

$$| w_n(x) - w_{n-1}(x) | \leqq (G(x) + F(x))^n, \tag{15}$$

where

$$G(x) = \int_x^\infty | f(s) |\, ds \tag{16}$$

(hence $G(x) \leqq F(x)$ if $x \geqq 1$, by (10)).

(iii) As a consequence of (ii), *the solution* $w(z)$, *assigned by* (11) *in* (i), *can be obtained by the process of successive approximations on some half-line* $x_0 < x < \infty$ (if x_0 is large enough, so large that, in (15),

$$G(x) + F(x) < 1 \text{ if } x_0 < x < \infty; \tag{17}$$

and (9) and the parenthetical remark made after (16) assure the existence of an x_0 satisfying (17)).

The refinement (iii) of (i), along with (ii) (where $x > x_0$ is *not* assumed) will be essential in what follows. The finite x-interval which is

debarred by the proviso $x > x_0$ of (iii) will prove to be such that its inclusion becomes possible if an appropriate use is made of (ii).

5. Now consider (2) under the assumption that, on the (open) half-plane $x > 0$, where x is the real part of $z = x + iy$, the coefficient function $f(z)$ is regular and representable as an absolutely convergent integral (1) (that is, that $f(z)$ is of class (A)), and identify (2) with (8), by keeping y fixed in $z = x + iy$, where

$$0 < x < \infty \tag{18}$$

and $-\infty < y < \infty$. Then what correspond to (12), (13) and (14) are

$$w_n(z) = 1 + \int_z^\infty (z - s)\{\int_0^\infty e^{-st} da_f(t)\} w_{n-1}(s)\, ds, \tag{19}$$

with

$$w_1(z) \equiv 0, \tag{20}$$

and

$$w_1(z), \cdots, w_n(z), \cdots. \tag{21}$$

Similarly, (15) becomes

$$|w_n(z) - w_{n-1}(z)| \leqq (G(x) + F(x))^n, \tag{22}$$

where, corresponding to (10) and (16),

$$F(x) = \underset{-\infty<y<\infty}{\text{l. u. b.}} \left| \int_z^\infty \left| s \int_0^\infty e^{-st}\, da_f(t) \right| ds \right| \tag{23}$$

and

$$G(x) = \underset{-\infty<y<\infty}{\text{l. u. b.}} \left| \int_z^\infty - \int_0^\infty e^{-st} da_f(t) \right| ds \Big|. \tag{24}$$

It follows from (i) that (2) will have a unique solution $w(z)$ satisfying the limit relation

$$w(z) \to 1 \text{ as } x \to \infty \tag{25}$$

(which is meant to be uniform for $-\infty < y < \infty$), if the condition corresponding to the convergence of the integral (10), where $x > 0$, is satisfied by f. This will certainly be the case if the expression (23) is finite at every point of the half-line $x > 0$. But it is clear that if (23) is replaced by

$$F(x) = \int_x^\infty)s \int_0^\infty e^{-st} \,|\, d\alpha_f(t) \,|\,) ds, \tag{26}$$

then (23) is finite if (26) is (at a given $x > 0$) and that, under the latter assumption,

$$G(x) = \int_x^\infty (\int_0^\infty e^{-st} \,|\, d\alpha_f(t) \,|\,) ds \tag{27}$$

is finite; finally, that (22) remains true if (as always in the sequel) the functions $F(x)$, $G(x)$ are defined by (26) and (27), rather than by (23), (24).

Accordingly, (i) and (ii) become applicable on (18) if (26) is convergent on (18). But (26), where x, s, t are positive, can be rearranged into

$$F(x) = \int_0^\infty e^{-xt}(1 + tx)t^{-2} \,|\, d\alpha_f(t) \,| \tag{28}$$

(since $t^2 \int_x^\infty se^{-st} ds = (1 + tx)e^{-st}$, and since Fubini's theorem on product measures is applicable whether the integrals are finite or not) and, correspondingly, (27) can be written in the form

$$G(x) = \int_0^\infty e^{-xt} t^{-1} \,|\, d\alpha_f(t) \,|. \tag{29}$$

On the other hand, since (1) is supposed to converge absolutely if $x > 0$, it is clear that (28) will be finite for every $x > 0$ if and only if (3) is satisfied (and that (29) is then certainly finite for every $x > 0$). But (3) is an assumption of the italicized theorem, to be proved.

Consequently, (i) and (ii) can be applied in the following proof of the theorem. In view of (i), it is already assured that (2) has a unique solution $w(z)$ satisfying (25). This solution exists, as a regular function, on the half-plane $x > 0$, since, by the local existence theorem of (analytic) linear differential equations, every solution of (2) is regular on every z-domain on which the coefficient function $f(z)$ is regular. What remains to be seen is that the particular solution $w(z)$, defined by (25), is a Laplace-Stieltjes transform, and that the latter is absolutely convergent on the entire half-plane $x > 0$ (and not, as one might expect from (iii), only on some half-plane $x > x_0$, where x_0 is sufficiently large).

6. To this end, (ii) will have to be used in the form obtained in Section 5, a form which states that (27), (26) and (19) satisfy (15) for $n = 1, 2, \cdots$ and at every point z of the half-plane $x > 0$.

It is clear from (ii) that each of the functions (21), defined by (19) and (20), is a regular function on the half-plane $x > 0$. This information will be completed by the following pair of facts:

(I) The sequence (21) is uniformly bounded on every fixed half-plane $x > \epsilon$, where $\epsilon > 0$.

(II) Each of the functions (21) is of class (A), that is, a Laplace-Stieltjes integral, say

$$w_n(z) = \int_0^\infty e^{-zt} d\beta_n(t), \tag{30}$$

which is absolutely convergent on the half-plane $x > 0$.

If (I) and (II) are granted, then it is possible to proceed as follows: According to (iii), there exists a half-plane $x > x_0$ on which the sequence (21) tends to the solution $w(z)$ determined by (25). But it follows from (I) that, in view of the theory of normal families, the "sufficiently large" x_0 can be improved to any $\epsilon > 0$. If this is combined with (II), it follows that the functions (21), besides being absolutely convergent integrals (30) on the half-plane $x > 0$, form a sequence which is uniformly bounded and (uniformly) convergent on every half-plane $x > \epsilon$, where $\epsilon > 0$ is arbitrary. But the assumptions placed on $f(z)$ in (2) are such as to remain satisfied if (2) is replaced by

$$w'' + f^*(z)w = 0 \quad \text{or} \quad w'' - f^*(z)w = 0,$$

where

$$f^*(z) = \int_0^\infty e^{-zt} \, |\, da_f(t) \,| \quad \text{for } x > 0;$$

cf. (1). Hence it is clear from (19) and (20) that not only the functions (30) but also the functions

$$w_n^*(z) = \int_0^\infty e^{-zt} \, |\, d\beta_n(t) \,|$$

form a sequence which is uniformly bounded and (uniformly) convergent on every half-plane $x > \epsilon$, where $\epsilon > 0$. It follows therefore from the "con-

tinuity theorem" of Laplace-Stieltjes transforms that, if $v(z)$ is the limit function of the sequence (30) (that is, if $v(z)$ denotes the solution $w(z)$ of (2) defined by (25)), then $v(z)$ is a Laplace-Stieltjes transform, say

$$v(z) = \int_0^\infty e^{-zt} d\beta(t), \tag{31}$$

which is absolutely convergent on the half-plane $x > 0$. In fact, if the constant $\beta_n(0)$, which (30) leaves undetermined for every n, is suitably chosen, then $w_n(z) \to v(z)$ as $n \to \infty$ (for $x > 0$) is equivalent to $\beta_n(t) \to \beta(t)$ as $n \to \infty$ in (30) and (31) (if the discontinuity points of $\beta(t)$ are disregarded).

Accordingly, the existence of the (unique) solution $w(z)$ which in (7) belongs to $(c_1, c_2) = (0, 1)$ follows if (I) and (II) are granted. On the other hand, if the existence of the solution $v(z)$ is granted, then the existence of the solution $zu(z)$, a solution $w(z)$ which belongs to $(c_1, c_2) = (1, 0)$ in (7), follows by the argument applied in [7], pp. 448-450. Consequently, the proof of the theorem will be complete if (I) and (II) are proved.

7. Clearly, each of the (complex) functions (19), where $z = x + iy$ and $x > 0$, is majorized by the corresponding (real, non-negative), function

$$W_n(x) = 1 + \int_x^\infty (s - x) \{ \int_0^\infty e^{-st} \mid da_f(t) \mid \} W_{n-1}(s)\, ds, \tag{32}$$

where $0 < x < \infty$ and, corresponding to (20),

$$W_1(z) \equiv 0. \tag{33}$$

Hence (I) will be proved if it is shown that the functions (32) are uniformly bounded on every closed half-line contained in the open half-line (18).

For similar reasons, (II) will follow if it is shown that each of the functions (32) is a Laplace-Stieltjes integral which is absolutely convergent on (18). Hence, somewhat more than (II) will be proved if it is shown that

(34) $W_n(x)$ is completely monotone on (18).

In fact, (34) and the Hausdorff-Bernstein theorem imply that if $\gamma_n(t)$ belongs to $W_n(x)$ in the same way as $\alpha(t)$ belongs to $f(z)$ in (1), then not only does there exist on (18) a Laplace-Stieltjes representation of $W_n(x)$ in terms of $\gamma_n(t)$ but, in addition, $d\gamma_n(t) \geqq 0$. But it also follows from

the Hausdorff-Bernstein theorem that (34) is equivalent to the following assertion:

(35_n) $\qquad (-D)^m W_n(x) \geqq 0$, where $m = 0, 1, \cdots$ and $0 < x < \infty$

$(D = d/dx)$. Hence (II) is true if $(35_1), (35_2), \cdots$ are true.

Since (35_1) is clear from (33), it follows that (II) will be proved if it is shown that (35_{n-1}) implies (35_n). This induction will be carried out now. The proof of (I) will be deferred to Section 8.

First, it is clear from Section 6 that each of the integrals (32) is convergent on (18). Next, (32) and a differentiation of (32) show that the cases $m = 0$ and $m = 1$ of (35_{n-1}) imply the cases $m = 0$ and $m = 1$ of (35_n). Hence, when proving that (35_{n-1}) implies (35_n), it is sufficient to consider the cases $m = 2, 3, \cdots$ of (35_n). But two differentiations of (32) show that

$$D^2 W_n(x) = W_{n-1}(x) \int_0^\infty e^{-xt} \, |\, da_f(t) \,|.$$

It follows therefore from (35_{n-1}), and from the fact that the product of two functions which are completely monotone on (18) is completely monotone on (18), that $D^2 W_n(x)$ is completely monotone on (18). Since this means that the cases $m = 2, 3, \cdots$ of (35_n) are true, the induction is complete.

8. In view of the end of Section 6 and of the beginning of Section 7, all that remains to be proved is that the functions (32) are uniformly bounded on every half-line $\epsilon < x < \infty$, where $\epsilon > 0$ is arbitrary.

Let (32) be written in the form

$$W_n(x) = 1 + \int_x^\infty (s - x) f^*(s) W_{n-1}(s) \, ds, \tag{36}$$

where

$$f^*(x) = \int_0^\infty e^{-xt} \, |\, da_f(t) \,|. \tag{37}$$

If Fubini's theorem is applied in the same way as in Section 6, then it is clear from (37) that (3) and the asbolute convergence of (3) and (18) together are equivalent to the (absolute) convergence of

$$\int_0^\infty t f^*(x + t) \, dt \qquad (f^* \geqq 0) \tag{38}$$

for every $x > 0$. Hence, the assertion is that (36), where $W_1(x) \equiv 0$, and the convergence of (38) on (18) imply the existence of a constant $C = C(\epsilon)$ satisfying

$$W_n(x) < C(\epsilon) \text{ for } \epsilon < x < \infty, \tag{39}$$

whenever $\epsilon > 0$. (All that is clear is that there exists a bound, depending on n, for which

$$W_n(x) < C_n(\epsilon) \text{ if } \epsilon < x < \infty; \tag{39 bis}$$

simply because each of the integrals (36) is convergent on (18).)

Since (36), where $W_1(x) \equiv 0$, implies that $W_2(x) \equiv 1$, it is clear that if

$$U_n(x) = W_n(x) - W_{n-1}(x), \tag{40}$$

then $U_1(x) \equiv 1$ and $U_n(x) = \int_x^\infty (s-x) f^*(s) U_{n-1}(s)\, ds$, that is,

$$U_n(x) = \int_0^\infty s f^*(x+s) U_{n-1}(x+s)\, ds. \tag{41}$$

The iteration of (41) shows that $U_{n+1}(x)$ is the n-fold integral

$$U_{n+1}(x) = \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{n} s_k f^*(x + s_1 + s_2 + \cdots + s_k)\, ds_k. \tag{42}$$

But since both (non-negative) integrals (37), (38) are convergent for every $x > 0$, it can readily be concluded from (42) that the functions $U_1(x)$, $U_2(x), \cdots$ are uniformly bounded on every fixed half-line $\epsilon < x < \infty$, where $\epsilon > 0$. In view of (40), where $W_1(x) \equiv 0$ and $W_2(x) \equiv 1$, this is equivalent to (39).

PART II.

Ordinary non-linear differential equations and Laplace transforms.

Introduction. The following considerations deal with three questions, (α), (β) and (γ), on families $x = x(t, c)$ of solutions of differential equations $dx/dt = f(t, x)$, where c is an integration constant, confined to a "sufficiently small" c-domain, and (A) if t is real, its range is either (a) a fixed interval which is not compact, say $0 \leqq t < \infty$, or (b) a half-line $T \leqq t < \infty$ with a $T = T_c$ which cannot be chosen independent of c, whereas (B) in the

analytic case, the t-range is either (a) a fixed half-strip ($0 \leqq \operatorname{Re} t < \infty$, $|\operatorname{Im} z| < \text{const.}$) or (b) a half-strip ($T \leqq \operatorname{Re} t < \infty$, $|\operatorname{Im} t| < \epsilon$) with $T = T_c$ and $\epsilon = \epsilon_c$.

In each of the four cases (Aa), (Ab), (Ba), (Bb), the three questions to be considered are (α) the existence of $x(t, c)$ on the infinite t-range, (β) the uniform continuity of $x(t, c)$ in t and c together (which, since the t-range is infinite, is equivalent to the question of stability in the sense of Dirichlet), finally (γ) the representation of $x(t) = x(t, c)$ as a Laplace-Stieltjes integral (the subcase (γA) of (γ) subsumes itself to the subcase (γB)). If (a_1), (a_2), (a_3) denote (α), (β), (γ) respectively, then the issue is the specification of conditions which, when satisfied by f, are sufficient for (a_i), where i is fixed. The resulting conditions will of course be stricter for (a_{i+1}) than for (a_i).

It will be sufficient to deal with the case in which x or f is a scalar. In fact, it will be clear from the nature of the proofs that the vector case (that is, the case of a system $dx_j/dt = f_j(t, x_1, \cdots, x_n)$, where $j = 1, \cdots, n$) can be reduced to the scalar case by a device of superordination, a device which reduces to the prinicple of majorization in the analytic case. Actually, the proofs are such as to make it clear that a direct treatment of the vectorial case differs from that of the scalar case in hardly more than in the intepretation of the symbols.

1. Suppose first that the f of

$$dx/dt = f(t, x) \tag{1}$$

is real-valued and continuous on the half $t \geqq 0$ of a real (t, x)-plane. Suppose further that

$$\int^{\infty} |f(t, 0)|\, dt < \infty, \tag{2}$$

and that there exists on the half-line $t \geqq 0$ a non-negative, continuous (but not necessarily bounded) function $\lambda(t)$ which satisfies the condition

$$\int^{\infty} \lambda(t) dt < \infty \tag{3}$$

and which is a "global Lipschitz factor" of f; that is, for which

$$|f(t, x') - f(t, x'')| \leqq \lambda(t) |x' - x''| \tag{4}$$

holds for any pair (t, x'), (t, x'') of points (t, x) contained in the half-plane

$$0 \leqq t < \infty, \qquad -\infty < x < \infty. \tag{5}$$

Under these assumptions, (1) is "unrestricted," that is, every solution $x = x(t)$ of (1) exists for

$$0 \leqq t < \infty. \tag{6}$$

What is more, every solution $x(t)$ of (1) tends, as $t \to \infty$, to a finite limit, $x(\infty)$. Conversely, if there is given any point c of the line

$$-\infty < c < \infty, \tag{7}$$

then there exists an initial condition x_0 corresponding to which the solution $x(t)$ determined $x(0) = x_0$ has the limit $x(\infty) = c$. These (and certain more general) facts were proved in [10]; for extensions, cf. the Appendix of this paper.

As pointed out to me by Professor Siegel (cf. [10], pp. 131-132), the correspondence between the $x(0)$-line and the $x(\infty)$-line is continuous and one-to-one in the general solution of (1), that is, by virtue of

$$x = x(t, x_0); \qquad x(0, x_0) = x_0, \qquad \lim_{t \to \infty} x(t, x_0) = x(\infty; x_0). \tag{8}$$

2. In what follows, it will be very essential that, under the assumptions (2)-(5), it is possible to choose a t^* in such a way that, on the portion

$$t^* \leqq t < \infty \tag{9}$$

of the half-line (6), the solutions $x(t) = x(t; c)$ *of* (1) can be obtained by the following process of sucessive approximations:

$$x_{n+1}(t) = c - \int_t^\infty f(s, x_n(s))\, ds, \tag{10}$$

where c is the arbitrary "initial constant" $x(\infty)$, the iteration process (10) starts with

$$x_0(t) \equiv c \qquad (c = x(\infty)), \tag{11}$$

and the convergence (in fact, even the absolute convergence) of each of the integrals (10) on the half-line (9) is part of the statement. In other words, the half-line (9) can be chosen in such a way that the sequence

$$x_0(t), x_1(t), \cdots, x_n(t), \cdots \tag{12}$$

will be defined by (11) and (10) and will converge on (9) (as a matter of fact, uniformly on (9)) to that solution $x(t)$ of (1) which belongs to $x(\infty)=c$. In fact, this was the "non-local" part of the proof in [10] (but not in [12], where nothing like a Lipschitz condition is assumed).

By adapting an argument which is well-known in the classical "local" problem (Bendixson-Lindelöf), it can be shown that (12) converges not only on a certain (constant) portion (9) of (6) but on the whole of (6), that is, for $0 \leqq t \leqq t^*$ as well. But this will not be needed in what follows.

3. In order to apply the (vectorial variants of the) facts collected in Section 1 to the problem of vortices and nodes (Poincaré), a simple device had to be applied in [14], pp. 821-823. In what follows, the adaptation of that device will be needed, in order to deal with the case in which the half-plane (5) of f is replaced by a strip

$$0 \leqq t < \infty, \qquad -b < x < b, \tag{13}$$

where $b>0$ is given arbitrarily (it can be "small").

Suppose that $f(t,x)$ is real-valued and continuous on (13) and satisfies (2) and (4), with (3), on (13). Then, corresponding to every positive b_0 which is less than b, there exist a sufficiently small $b^*=b^*(b_0)>0$, a sufficiently small $c^*=c^*(b_0)$ and a sufficiently large $t^*=t^*(b) \geqq 0$ which have the following properties: Every solution $x(t)$ of (1) which belongs to any initial condition $x(t^*)=x^*$ satisfying

$$|x^*| < b^* \qquad (x^*=x(t) \text{ at } t=t^*) \tag{14}$$

exists on (9), satisfies

$$|x(t)| < b_0 \tag{15}$$

on (9), and tends, as $t \to \infty$, to a limit c satisfying

$$|c| < c^* \qquad (c=x(\infty)). \tag{16}$$

Conversely, there belongs to every c satisfying (16) a unique solution $x(t)$ of (1) which exists on (9), satisfies the inequality (15) on (9) and the conditions $c=x(\infty)$ and (14), where $x^*=x(t^*)$ (note that $b^*>0$ can be chosen arbitrarily small; in particular, so as to be less than b_0). Finally, for every c satisfying (16), not only (15) but also

$$|x_n(t)| < b_0 \qquad (n=0,1,\cdots) \tag{17}$$

holds on (9), where the functions (12) are defined, on (9), by (11) and

(10) and converge, on (9), to that solution $x(t)$ of (1) which is determined by the "initial condition" $x(\infty) = c$.

First, since $0 < b_0 < b$, and since $f(t,x)$ is given on (13) as a continuous function satisfying (2) and (4), with (3), it is clear that if (13_0) denotes the strip which results if b is replaced by b_0 in (13), then it is possible to choose on the half-plane (5) a continuous $f(t,x)$ which is identical with the given $f(t,x)$ on the portion (13_0) of (13) and which satisfies (2) and (4), with (3), on (5). Hence, all the assertions, concerning the existence of the positive numbers b^*, c^*, t^* which have the above-stated properties, follow from Sections 1-2 and from the very nature of the process of the sucessive approximations (cf. [10]).

4. It is also clear from the nature of the successive approximations that nothing is changed if the real differential equation (1) is replaced by the complex differential equation

$$dw/dz = f(z,w), \tag{18}$$

where, corresponding to the continuity of $f(t,x)$ on the strip (13), the function $f(z,w)$ is supposed to be regular on

$$\operatorname{Re} z > 0, \qquad |w| < b, \tag{19}$$

the product space of a circle $|w| < b$ and of the half-plane $x > 0$, where $z = x + iy$, $-\infty < y < \infty$. Then there exists on (19) an absolutely convergent expansion

$$f(z,w) = \sum_{m=0}^{\infty} f_m(z)\, w^m, \tag{20}$$

where every

$$f_m(z) \text{ is regular for } x > 0 \qquad (x = \operatorname{Re} z). \tag{21}$$

What corresponds to (2) is the condition

$$\int_{-\infty}^{\infty} \text{l. u. b.}_{-\infty<y<\infty} |f_0(x+iy)|\, dx < \infty. \tag{22}$$

Similarly, what corresponds to (4) and (3) on (13) is certainly satisfied if

$$\int_{-\infty}^{\infty} \text{l. u. b.}_{-\infty<y<\infty} h(x+iy)\, dy < \infty, \text{ where } h(z) = \text{l. u. b.}_{|w|<b} |f(z,w)/w)| \tag{23}$$

(provided that the integrand of (23) is continuous (and $\neq \infty$) at every x). Incidentally, less than this proviso would suffice in (23); cf. [13]).

These are all the assumptions which correspond to those of Section 3. Hence, under these assumptions, there will belong to every circle $|z| < b_0$, where $0 < b_0 < b$, a half-plane $\mathrm{Re}\, z > d^*$, a circle $|w_0| < b^*$ $(\leqq b_0)$, where $w_0 = w(z)$ at $z = z_0$ and $\mathrm{Re}\, z_0 > d^*$, and a circle $|c| < c^*$, where $c = w(\infty)$, in such a way that the assertions of Section 3 hold (in their complex version). It is understood that $w(\infty) = c$ now means that $w(x+iy) \to c$ holds uniformly for $-\infty < y < \infty$ as $x \to \infty$.

5. As an application, choose on the half-line

$$0 \leqq s < \infty \tag{24}$$

a sequence of functions $a_0(s), a_1(s), \cdots$ satisfying $[a_m] < \infty$, where $m = 0, 1, \cdots$ and

$$[a] = \int_0^\infty |da(s)|. \tag{25}$$

Then (21) is certainly satisfied if

$$f_m(z) = \int_0^\infty e^{-zs} da_m(s), \tag{26}$$

and (20) is certainly regular on (19) if

$$\sum_{m=0}^{\infty} [a_m] b^m < \infty, \text{ where } [a_m] = \int_0^\infty |a_m(s)|. \tag{27}$$

Actually, these assumptions are much stricter than necessary; suffice it to say that, for every fixed m, the integral (26) can have a finite abscissa of boundedness and uniform convergence without having a finite abscissa of absolute convergence.

In view of the case $m = 0$ of (26), and since

$$\int_1^\infty s^{-1} |da(s)| \leqq \int_1^\infty |da(s)| \leqq [a] < \infty,$$

condition (22) is certainly satisfied if

$$\int_0^1 s^{-1} |da_0(s)| < \infty. \tag{28 bis}$$

On the other hand, since it is clear from (20) and (26) that the function $h(z)$ occurring in (23) is

$$h(z) = \text{l. u. b.}_{|w|<b} \Big| \sum_{m=1}^{\infty} m w^{m-1} \int_0^{\infty} e^{-zs} da_m(s) \Big| \leqq \sum_{m=1}^{\infty} m b^{m-1} \int_0^{\infty} e^{-xs} \, | da_m(s) |,$$

condition (23) is certainly satisfied if

$$\sum_{m=1}^{\infty} m b^m \int_0^1 s^{-1} \, | da_m(s) | < \infty.$$

Hence, if the notation is changed so as to replace the given $b > 0$ by a somewhat greater $b > 0$, it is clear that (23) is satisfied if

$$\sum_{m=1}^{\infty} b^m \int_0^R s^{-1} \, | da_m(s) | < \infty$$

holds for $R = \infty$ or, what in view of (27) is the same thing, for $R = 1$. Finally, since $m > 0$ in the last sum, it follows that not only condition (23) but also condition (22) (which belongs to $m = 0$ and was taken care of by (28 bis) above) will be satisfied if, in addition to (27),

$$\sum_{m=0}^{\infty} \{a_m\} b^m < \infty, \text{ where } \{a_m\} = \int_0^1 s^{-1} \, | da_m(s) |. \tag{28}$$

[Note that $a_m(+0) = a_m(0)$, where $m = 0, 1, \cdots$, is a necessary condition for (28).]

Accordingly, the situation is as follows: If the coefficient functions of (20) are of the form (26) and satisfy (27) and (28) for some $b > 0$ (that is, if $[a_m]^{1/m}$ and $\{a_m\}^{1/m}$, where $m = 0, 1, \cdots$ and $1/0 = 1$, are less than a constant), then the results of Section 4 are applicable to (18).

This implies, among other things, that there will exist a sufficiently large $d^* > 0$ and a sufficiently small $b^* > 0$ in such a way that, whenever z_0 and w_0 are numbers satisfying

$$\text{Re}\, z_0 > d^* \text{ and } | w_0 | < b^* \tag{29}$$

respectively, the initial condition

$$w(z_0) = w_0 \tag{30}$$

will determine a solution

$$w = w(z) = w(z; z_0, w_0) \tag{31}$$

of

$$dw/dz = \sum_{m=0}^{\infty} w^m \int_0^{\infty} e^{-zs} da_m(s) \tag{32}$$

which exists (as a regular function) on the half-plane

$$\operatorname{Re} z > \operatorname{Re} z_0, \tag{33}$$

and each of these solutions (31) will tend to a finite limit $c = c(z_0, w_0)$:

$$w(z) \to c \text{ as } \operatorname{Re} z \to \infty. \tag{34}$$

In addition, these solutions $w(z)$ can be obtained (if b^* and $1/d^*$ are small enough) by the following process of successive approximations:

$$\lim_{n\to\infty} w_n(z) = w(z) \text{ on (33)}, \tag{35}$$

where the sequence

$$w_0(z), w_1(z), \cdots, w_n(z), \cdots \tag{36}$$

is defined, corresponding to (11) and (12), by

$$w_{n+1}(z) = c - \int_z^{\infty} \sum_{m=0}^{\infty} f_m(\zeta) (w_n(\zeta))^m d\zeta \qquad \text{[cf. (26)]} \tag{37}$$

and

$$w_0(z) \equiv c \qquad (c = w(\infty)) \tag{38}$$

together, and the integrals (37) are convergent on (33) and represent regular functions satisfying

$$|w_n(z)| < b \text{ on (33)}; \tag{39}$$

cf. (17).

6. On this basis, the following theorem will now be proved:

Let $f_0(z), f_1(z), \cdots$, where $\operatorname{Re} z > 0$, be a sequence of functions (26) *satisfying* (27) *and* (28) *for some $b > 0$, and let $f(z, w)$ be defined by* (20) *on the domain* (19). *Suppose further that $f_0(z) \equiv 0$. Then there exist two positive numbers, d^* and b^* ($\leqq b$), having the following properties: If z_0 and w_0 are complex numbers within the circle $|z_0| < b^*$ and the half-plane*

$\operatorname{Re} z_0 > d^*$ *respectively, then the solution* $w = w(z) = w(z; z_0, w_0)$ *of the (local) initial-value problem*

$$dw/dz = f(z, w), \qquad w(z_0) = w_0$$

exists on the half-plane $\operatorname{Re} z > d^*$ *and is representable as a Laplace-Stieltjes integral*

$$w(z) = \int_0^\infty e^{-zs} d\beta(s) \tag{40}$$

which is absolutely convergent for $\operatorname{Re} z > d^*$; *in addition,*

$$|w(z)| < b \text{ for } \operatorname{Re} z > d^*. \tag{40 bis}$$

Needless to say, the function $\beta(s)$ occurring in (40) depends on the initial constant $w(z_0) = w_0$ (and is of bounded variation on every finite interval of the half-line $0 \leqq s < \infty$).

Theorems of this type are known under substantially stricter assumptions. In fact, the known results deal either with linear differential equations (cf. [8]) or, in the non-linear case typified by (32), assume (cf. [9]) that

$$a_m(s) \equiv 0 \text{ for } 0 \leqq s \leqq 1, \text{ where } m = 0, 1, \cdots \tag{41}$$

(if the unit of length is suitably chosen on the half-line (24) and if the functions $a_m(s)$, in which (26) leaves undetermined an additive constant, are normalized by $a_m(1) = 0$). But (41) is a very drastic way of satisfying (28), whereas (if (27) is assumed) the allowable behavior of the functions $a_m(s)$ *near* $s = 0$ is the main issue. In fact, what is involved is a problem in "small divisors" (cf. Part I, Section 2, above).

On the other hand, (27) and (28) are the true conditions. In order to see this, it is sufficient to consider the particular case in which (32) becomes homogeneous and linear, that is,

$$dw/dz = f(z)w, \text{ with } f(z) = \int_0^\infty e^{-zs} da(s), \tag{42}$$

where $a_m \equiv 0$ if $m \neq 1$ and $a_1 = a$. Then (27) and (28) reduce to

$$\int_0^\infty |da(s)| < \infty \text{ and } \int_0^1 s^{-1} |da(s)|\, ds < \infty. \tag{43}$$

But if $da(s) \geqq 0$ (that is, if $da = |da|$), then a quadrature shows that (42) cannot possess any solution $w(z) \not\equiv 0$, representable for sufficiently large

$\operatorname{Re} z$ as an (absolutely) convergent integral (40), unless both conditions (43) are satisfied.

The proof for the sufficiency of (27) and (28) seems to call for elaborate estimates of the total variations of the successive approximations to the (unknown) function $\beta(s)$ which occurs in (40). It turns out however that this elaborate work (which, in my first attempts to prove the sufficiency of (27) and (28), appeared to be too involved to be completed by explicit induction) can be avoided entirely. This will be made possible by an appeal to the "continuity theorem" of the transform (40) (cf. the end of Section 2 in Part I).

7. It will be clear from the proof of the italicized theorem that, for reasons of majorization, it is sufficient to prove the theorem for the case in which the functions $a_m(s)$ occurring in (26), (27), (28) are real-valued and non-decreasing (so that $da_m(s) = |da_m(s)|$ for $m = 0, 1, \cdots$ and $0 \leqq s < \infty$) and, in addition, the circle $|w_0| < b^*$ and the half-plane $\operatorname{Re} z_0 > d^*$ of the initial conditions are replaced by the interval $0 \leqq w_0 < b^*$ and the half-line $d^* < z_0 < \infty$ respectively. For the same reason, it will be sufficient to consider the differential equation (32) and the existence of its solution (40) only on the half-line $d^* < x < \infty$, where $x = \operatorname{Re} z$ (correspondingly, z_0 will be denoted by x_0). Then it is clear from the way in which the results of Section 5 were obtained that $c \geqq 0$ in (34).

The c of (34) is the same as the c of the recursion formulae (37)-(38). Since $da_m(s) = |da_m(s)|$ in (26), these recursion formulae now become

$$w_{n+1}(x) = c - \int_x^\infty \sum_{m=0}^{\infty} \left(\int_0^\infty e^{-ts} |da_m(s)| \right) (w_n(t))^m dt, \tag{44}$$

where $0 < d^* < x_0 \leqq x < \infty$ and

$$w_0(x) \equiv c \geqq 0 \qquad (c = w(\infty)). \tag{45}$$

If d^* and b^* are so chosen as in Section 5 (in connection with (29)), and if c is small enough (with reefrence to d^* and b^*), then, according to Sections 4-5, all integrals (44) (which in view of (45) have non-negative integrands) exist ($< \infty$) on the half-line

$$x_0 < x < \infty \tag{46}$$

and satisfy the limit relation

$$\lim_{n \to \infty} w_n(x) = w(x) \text{ on (46), where } w(\infty) = c, \tag{47}$$

where $w(x)$ is a solution of $dw/dx = f(x, w)$. Hence, in view of the concluding sentence of Section 6, the italicized theorem will be proved if it is shown that each of the functions $w_n(x)$ is representable in the form

$$(48) \qquad w_n(x) = \int_0^\infty e^{-xs} d\beta_n(s) \text{ on } (46), \text{ where } d\beta_n(s) \geqq 0.$$

But this can be shown by an adaptation of the corresponding step in Part I.

In fact, the assertion of (48) is equivalent to the statement that each of the functions $w_n(x)$ is completely monotone on (46), that is, that, if $D = d\ /dx$,

$$(49_n) \qquad (-D)^k w_n(x) \geqq 0 \text{ on } (46), \text{ where } k = 0, 1, \cdots$$

(Hausdorff-Bernstein). Since the truth of (49_0) is clear from (45), it is sufficient to prove that (49_{n+1}) is implied by (49_n). But (45) and the case $n = 0$ of (44) show that the case $k = 0$ of the inequalities (49_n) is true for every n. On the other hand, differentiation of (44) shows that

$$(50) \qquad -Dw_{n+1}(x) = \sum_{m=0}^{\infty} \left(\int_0^\infty e^{-xs} \mid da_m(s) \mid \right) (w_n(x))^m.$$

Since $-Df(x)$, $g_1(x) + g_2(x)$ and $\lim f_m(x)$, where $m \to \infty$, are completely monotone on (46) if $f(x)$, $g_j(x)$ and $f_m(x)$ are (provided that $\lim f_m(x)$ exists), it follows by successive differeniations of (50) that (49_n) is true for every n and every m if every (49_n) is true for $m = 0$ (cf. the corresponding induction in Section 7 of Part I).

8. Accordingly, all that remains to be verified is the case $k = 0$ of (49_n) for every n, that is,

$$(51_n) \qquad w_n(x) \geqq 0 \text{ on } (46).$$

Since (51_0) is clear from (45), it is sufficient to prove that (51_{n+1}) is implied by (51_n). This induction will prove (49_n) for every n if $c > 0$ in (45). The limiting case $c = 0$ will have to be discussed separately.

Suppose first that $c > 0$. Then, since the integral occurring in (44) (for a fixed n) is convergent for $d^* < x < \infty$ (where $d^* < x_0$), it is clear from (44) that $w_{n+1}(x) > 0$ is true for every x which is large enough with reference to c and $n + 1$. But (50) holds on the *fixed* half-line (46), since $x_0 > d^*$. Since the truth of (51_n) is assumed (for a fixed n), it follows that $Dw_{n+1}(x) \leqq 0$ on (46). Consequently, (51_{n+1}) follows from the fact that

$w_{n+1}(x) \geqq 0$ is true for *large* x (it also follows that the sign of equality cannot take place in (51_n) for any n).

In the remaining case, $c = 0$. But the assumption $f_0(z) \equiv 0$ of the italicized theorem (Section 6), an assumption which reduces (20) to

$$f(z, w) = \sum_{m=1}^{\infty} f_m(z) w^m, \tag{52}$$

was not used thus far. If it is used, then it is clear that (18) admits the solution $w(z) \equiv 0$. Since the latter belongs to the case $c = 0$ of (34) and can be represented in the form (40), with $d\beta(s) \equiv 0$, the proof is complete.

9. Since the assumption that (20) reduces to (52) was used nowhere but in the last paragraph of Section 8, the following theorem was also proved above:

Let $f_0(z), f_1(z), \cdots$, *where* $\mathrm{Re}\, z > 0$, *be a sequence of functions* (26) *satisfying* (27) *and* (28) *for some* $b > 0$. *Then, if* $r = r(b) > 0$ *is sufficiently small and if* c *denotes any complex number satisfying* $0 < |c| < r$, *there exists an abscissa* $d = d(c) < \infty$ *having the following properties: The differential equation* (18) *has a solution* $w(z) = w(z; c)$ *which exists on the half-plane* $\mathrm{Re}\, z > d$, *satisfies the "initial condition"* (34), *and is representable on the half-plane* $\mathrm{Re}\, z > d$ *as an absolutely convergent Laplace-Stieltjes integral* (40), *where* $\beta(s) = \beta_c(s)$ (*and* $w(z)$ *satisfies the inequality* $|w(z)| < b$ *if* $\mathrm{Re}\, z > d$).

Since $c = 0$ is excluded, it is clear from (34) and (40) that $\beta(s) = \beta_c(s)$ has a jump $(= c = \beta(+0) - \beta(0) \neq 0)$ at $s \neq 0$.

APPENDIX.

On a non-local theory of the initial value problem of ordinary differential equations.

1. Let H_0 denote the open half-line $0 < t < \infty$, H the closure of H_0, R the n-dimensional x-space $(-\infty < x_i < \infty\ ;\ i = 1, \cdots, n)$, with $x = (x_1, \cdots, x_n)$ and $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$, and $f(t, x)$ a vector function, with n real components, which is defined and continuous (but not necessarily uniformly continuous or bounded) on $H \times R$. According to Peano,

$$dx/dt = f(t, x) \tag{1}$$

defines a (local) flow. The state of R under this flow at the date t will be denoted by R_t; so that R_0 is the space R of all initial conditions x_0 which can be assigned for a (not necessarily unique) solution $x(t)$ of (1) as $x(0) = x_0$. Eventually, under appropriate restrictions of the given function $f(t,x)$ (cf. (C) below), even the state R_∞ of R will be considered.

In [10]-[13] I investigated various non-local issues concerning the solutions $x = x(t;x_0)$ of (1) (when f is subject to appropriate restrictions); that is, issues pertaining to the existence and behavior of $x(t;x_0)$ on the whole of H (and for any point x_0 of R), rather than just on some "sufficiently small" t-interval. The main questions can be listed as follows:

Type (A). Every solution $x(t;x_0)$ of (1) should be "unrestricted," that is, such as to be "continuable" for $0 \leqq t < \infty$ (the standard example in which such is *not* the case is Riccati's scalar differential equation $x' = x^2$, possessing the solution $x(t) = -(t-t_0)^{-1}$ which exists for $0 \leqq t < t_0$ if $0 < t_0 < \infty$ but cannot be extended into any solution for $0 \leqq t < t^0$ if $t^0 > t_0$).

Type (A*). Let (1) be of type (A) and such as to possess the property that $\limsup |x(t;x_0)| < \infty$, where $t \to \infty$, holds for every solution ("stability" for every fixed x_0). A refinement of this type is

Type (B). Let (1) be of type (A) and such as to possess the property that, corresponding to every solution $x(t;x_0)$, there exists a point $c = c(x_0)$ of R satisfying $x(t;x_0) \to \infty$ as $t \to \infty$ (this will be expressed by saying that $x(\infty;x_0)$ exists on R_0).

Type (B*). Let (1) be of type (A) and such that there exists, as $t \to \infty$, a finite $\lim |x(t)|$ for the *length* of every solution under $x(t)$ of (1) (this condition, concerning *amplitudinal* limits, requires more than (A*) but less than (B)).

Type (C). Let (1) be of type (B) and such as to possess the property that every point of R is the $x(\infty;x_0)$ of at least one point x_0 of R_0 (so that one can speak of R_∞).

Type (D). Let (1) be of type (C) and such as to possess the property that the correspondence between R_0 and R_∞, that is, the mapping $x_0 \to x(\infty;x_0)$, is continuous and one-to-one.

Type (D bis). Let (1) be of type (A) and such as to possess the property that $x(t;x_0) \to 0$, as $t \to \infty$, holds for every solution. This type is

of type (B) but represents just the opposite of type (C). A variant of this "confluent" type (in which the point, 0, of "confluence" could of course be replaced by any point of R) is

Type (D*). Let (1) be of type (A) and such as to possess the property that $|x(t;x_0)| \to 0$, as $t \to \infty$, holds for every solution (this type is just the opposite of type (A*)).

The issue is the specification of conditions which, when satisfied by the given function $f(t,x)$, assure that (1) belongs to one of these types. Such criteria were given *loc. cit.* In what follows, the results available to the methods applied there will be systematized and further developed.

2. The starting point for any non-local statement (that is, of a statement on H) must be the following lemma (cf. [11], and [18], p. 177):

LEMMA. *If there is given for* (1), *where* $f(t,x)$ *is continuous on* $H \times R$, *a solution* $x(t)$ *on an interval* $0 \leqq t < t_0$, *where* $t_0 < \infty$, *then either there exists some interval* $0 \leqq t < t^0$, *where* $t^0 > t_0$, *on which* (1) *possesses some solution which is identical with the given solution for* $0 \leqq t < t_0$ *or else* $|x(t)| \to \infty$ *holds as* $t \to t_0 - 0$ [in which case the "non-continuability" of the given solution, that is, the non-existence of any $t^0 > t_0$, is clear from the fact that a function $x(t)$ cannot satisfy (1) at $t = t_0$ if it is not differentiable at t_0; hence certainly not if it fails to be bounded near t_0].

The alternative implies that $\liminf |x(t)| < \infty$, where $t \to t_0 - 0$, assures the existence of $\lim |x(t)|$ and, what is more, the existence of $\lim x(t)$; and that $\limsup |x(t)| = \infty$ cannot hold unless $\lim |x(t)| = \infty$. The latter point is the crucial step in the proof of the Lemma, which in [11] I abstracted from Painlevé's (and Weierstrass') considerations on the singularities of the problem of three bodies (Weierstrass's investigations are of an earlier date, but were published, posthumously, much later, than those of Painlevé; cf. [16] and [17]).

By a generalization of the argument applied in the Appendix of [11], a generaliaztion suggested by a local criterion of Kamke [15], pp. 139-141, the following comparison theorem can be derived from the Lemma:

(i) *Let* $f(t,x)$ *be continuous on* $H \times R$ *and, if* H_0 *and* J *denote the open half-line* $0 < t < \infty$ *and the closed half-line* $0 \leqq r < \infty$ *respectively, let there exist on* $H_0 \times J$ *a continuous function* $g(t,r)$ *satisfying*

$$|f(t,x)| \leqq g(t,|x|) \text{ on } H_0 \times R \qquad (g \geqq 0) \tag{2}$$

and having the following property: If any solution $r=r(t)$ *of*

$$dr/dt=g(t,r) \tag{3}$$

is given on any t-interval I contained in H_0 *(with* $(t,r(t))$ *in* $I\times J$*), then there exists on* H_0 *a function* $r(t)$ *(with* $(t,r(t))$ *in* $H_0\times J$*) which is a solution of* (3) *on* H_0 *and reduces to the given* $r(t)$ *on I. Then* (1) *is of type* (A).

The proof of this (A)-criterion will be such as to supply the following (A*)-criterion also:

(i*) *Let f and g satisfy the assumptions of* (i). *Suppose further that* $\limsup r(t)<\infty$, *where* $t\to\infty$, *holds for every solution* $r(t)$ $(\geqq 0)$ *of* (3). *Then* (1) *is of type* (A*).

The assertion of the Lemma is that any solution $x(t)$ of (i), given for small $t-t_0\geqq 0$, where t_0 $(\geqq 0)$ is fixed, can be extended to a solution of (1) on H. It will be sufficient to prove the existence of such an extension for $t_0\leqq t<\infty$. Suppose that there does not exist such an extension. Then, by the Lemma, there is a finite $t^0>t_0$ having the property that $x(t)$ exists, but is not bounded, on the interval $t_0\leqq t<t^0$. Consider a solution $r=r(t)$ of (3) belonging to the initial condition $r(t_0)=r_0$, where $r_0=|x(t_0)|$. According to the assumption made in (i) on (3), the solution $r(t)$ will exist on H_0 and will, therefore, be bounded on $t_0\leqq t<t^0$. But (1), (2), (3) and $|x(t_0)|=r(t_0)$ imply that $|x(t)|\leqq r(t)$ for $t_0\leqq t<t^0$; hence $x(t)$ remains bounded as $t\to t^0-0$. This contradiction proves (i).

It is also seen that $|x(t)|\leqq r(t)$ for $t_0\leqq t<\infty$. Since this inequality holds whether the additional assumption of (i*) is satisfied or not, (i*) follows.

A particular useful corollary of (i), a corollary in which the assumptions which (i) posits for $g(t,r)$ become replaced by "explicit" assumptions, is the following:

(i_0) *Let* $f(t,x)$ *be continuous on* $H\times R$ *and let there exist, on the respective open half-lines* $(0<t<\infty)$, $(0<r<\infty)$, *two positive, continuous functions* $\lambda=\lambda(t)$, $\phi=\phi(r)$ *satisfying*

$$|f(t,x)|\leqq\lambda(t)\phi(r) \text{ on } H_0\times R \tag{4}$$

and

$$\int_1^\infty dr\,/\phi(r)=\infty. \tag{5}$$

Then (1) *is of type* (A).

This can be reduced to the criterion proved in the Appendix of [11] if the dt of (1) is replaced by $\lambda(t)dt$ and, when the old t in $f(t,x)$, where $f=(f_1,\cdots,f_n)$ and $x=(x_1,\cdots,x_n)$, is denoted by x_0, the n-ary system (1) is replaced by the $(n+1)$-ary system $dx^*/dt=f^*(x^*)$, where $x^*=(x_0,x)$ and $f^*=(1,f)$. Actually, (i_0) is a particular case of (i). In fact, if $g(t,r)=\lambda(t,r)$, then (2) and (3) become (4) and

$$dr/dt=\lambda(t)\phi(r) \tag{6}$$

respectively. But if (6) is written in the form $dr/\phi(r)=\lambda(t)dt$ (when $r>0$, hence $\phi(r)>0$), and if it is observed that $\lambda(t)$ stays between positive bounds on compact subintervals of H_0, then it is seen from (5) that the condition required of (3) in (i) is satisfied by (6).

3. The (A)-criterion (i_0) has the following (B)-counterpart:

(ii) *If the assumptions of* (i_0) *are satisfied and if, in addition,*

$$\int_1^\infty \lambda(t)dt<\infty, \tag{7}$$

then (1) *is of type* (B).

In fact, (ii) can be concluded from (i_0) in about the same way as (i_0) was concluded from (i); cf. [12].

A (C)-counterpart of (i) is as follows:

(iii) *If the assumptions of* (ii) *are satisfied and if, in addition,*

$$|f(t,x')-f(t,x'')|\leqq\lambda(t)\phi(|x'-x''|) \text{ on* } H_0\times R'\times R'' \tag{8}$$

and

$$\int_{+0}^\infty |f(t,0)|\,dt<\infty, \tag{9}$$

then (1) *is of type* (C).

This was proved in [13]. Actually, (iii) can be concluded from (ii) in about the same way as (ii) is concluded from (i_0).

(iii°) *If the assumptions of* (iii) *are satisfied and if, in addition,* (7) *is supplemented by*

$$\int_\varepsilon^1 \lambda(t)dt<\infty, \tag{10}$$

* By R' and R'' are meant the x-space R when $x=x'$ and $x=x''$ respectively.

and (5) *by*

$$\int_{\epsilon}^{1} dr\ /\phi(r) = \infty, \tag{11}$$

finally (8) *by*

$$|f(0,x') - f(0,x'')| \leqq \phi(|x' - x''|) \text{ on } R' \times R'', \tag{12}$$

then there belongs to every point (t_0, x_0) *of* $H \times R$ *only one solution* $x(t)$ *of* (1) *satisfying* $x(t_0) = x_0$.

In fact, what (iii°) adds to (iii) is a purely local statement, and the latter is nothing but Osgood's well-known criterion for local uniqueness; cf. [15], p. 100 and p. 142. On the other hand, something very different from this local assertion is contained in the following (D)-counterpart of the (C)-criterion (iii):

(iv) *If the assumptions of* (iii°) *are satisfied, then* (1) *is of type* (D).

It is clear from the definition of the (D)-type that the truth of (iv) presupposes the truth of (iii°).

The case $\phi(r) = r$ of (iv) was concluded by Siegel (cf. [10], pp. 131-132) from the proof of the case $\phi(r) = r$ of (iii) (cf. [10], pp. 125-131) by a simple topological argument. But a glance at that argument shows that nothing is changed if $\phi(r) = r$ is replaced by any $\phi(r)$ satisfying (5) and (11).

4. The (B*)-character of (1) (that is, the existence of finite *amplitudinal* limits) can be assured by an assumption on f which, in contrast to the preceding "bilateral" assumptions, is just "unilateral":

(ii*) *Let* $f(t,x)$ *be continuous on* $H \times R$ *and suppose that*

$$x \cdot f(t,x) \leqq 0 \text{ on } H_0 \times R, \tag{13}$$

(*where* $x \cdot f = x_1 f_1 + \cdots + x_m f_m$) *or, more generally, that there exists on* H_0 *a non-negative* $\lambda(t)$ *satisfying*

$$\int^{\infty} \lambda(t)\,dt < \infty \text{ if } \epsilon > 0 \tag{14}$$

and

$$(x \cdot f(t,x))^+ \leqq \lambda(t) \min(1, |x|^2) \text{ on } H_0 \times R \tag{15}$$

(*where* $(x \cdot f)^+ = \max(0, a)$ *for* $a = x \cdot f$). *Then* (1) *is of type* (B*).

Remark. The choice of the term $\phi(|x|^2) = |x|^2$ in the factor min of $\lambda(t)$ in (15) corresponds to the particular case $\phi(r) = r$ in the preceding criteria. Correspondingly, it will be clear from the proof of (ii*) how to extend (ii*) to other choices of $\phi(|x|^2)$. The choice $\phi(|x|^2) = |x|^2$, made in (ii*) for the sake of simplicity, is important in the theory of *linear* systems (1); cf. [13], pp. 558-559, or [19].

In order to prove (ii*), put $\frac{1}{2}|x|^2 = s = s(t)$ along any solution $x = x(t)$ of (1) which is given on an interval $0 \leqq t < t_0$. Consider first the particular case (13) of (15) (cf. [13], p. 557). Then scalar multiplication of (1) by x shows that $ds(t) \leqq 0$ if $0 \leqq t < t_0$. Hence $s(t_0 - 0) = \infty$ is impossible if $t_0 < \infty$, and so the Lemma assures that t_0 can be assumed to be ∞; so that (1) is of type (A). But then it is clear from $ds(t) \leqq 0$ and $s(t) \geqq 0$ that there must exist a finite $s(\infty) \geqq 0$; that is, that there must exist for $x(t)$ an amplitudinal limit, which is the (B*)-assertion of (ii*).

Next, replace (13) by (15) but suppose, in addition to (15), that

$$x \cdot f(t \cdot x) \geqq 0 \text{ on } H_0 \times R. \tag{16}$$

Then it follows from (1) that, if $0 \leqq t < t_0$ (where, for the present, $t_0 = \infty$ *or* $t_0 < \infty$), then

$$0 \leqq ds(t)/dt \leqq \lambda(t) \min(1, s(t)). \tag{17}$$

Since $ds/s = d \log s$, it is clear from (17) and (14) that there cannot exist a finite t_0 satisfying $s(t_0 - 0) = \infty$, and so, for the same reason as before, (1) must be of type (A). But then, by letting $t \to \infty$ in (17), and using $ds/s = d \log s$ and (14) again, one is led to the existence of a finite $s(\infty) > 0$, provided that $s(t) = \frac{1}{2}|x(t)|^2$ does not vanish for large t.

If $s(t) > 0$ only on a sequence of (open) t-intervals $I_1, I_2, \cdots$ which cluster at $t = \infty$, but $s(t) = 0$ on the (closed) set K complementary to $L = I_1 + I_2 + \cdots$, then the existence of a finite $\lim s(t)$ follows if t tends to ∞ so as to be confined to $I_1^* + I_2^* + \cdots = L^* = L^*(\epsilon_1, \epsilon_2 \cdots)$, where $I_n^* = I_n^*(\epsilon_n)$ denotes, for any positive $\epsilon_n < 1$, an interval contained in I_n and having a length $|I_n^*| < (1 - \epsilon_n)|I_n|$. But the value of $\lim s(t)$, with t on L^*, is independent of the choice of L^*, since L^* is in L for every choice of $\epsilon_1, \epsilon_2, \cdots$. Hence, if every ϵ_n is let to tend to 0 (in an appropriate way), then, since $s(t) = 0$ when t is an end-point of an I_n, it is seen that $s(t) \to 0$ holds when t tends to ∞ on L. Since $s(t) = 0$ on the complementary set K, this means that $s(\infty)$ exists and is 0.

There remains to be considered the general case (15), where neither (13) nor (16) is assumed. But this case can be reduced to the treatment

of the preceding two cases, if the device of splitting the t-axis into an open set L and a closed set K is now chosen as follows: t is in L or in K according as $ds(t)/dt > 0$ or $ds(t)/dt \leqq 0$. The assertion of (ii*) then follows.

The definition of the "confluent" type (D bis) (formulated after the definition of type (D) above) requires that $s(\infty)$ or $\lim |x(t)|$ should not only exist (as a finite limit which may be positive or 0) but should be 0, for every solution $x(t)$ of (1). A corresponding criterion is supplied by the following modification of the preceding (B*)-criterion (ii*):

(ii* bis) *Let $f(t,x)$ be continuous on $H \times R$ and suppose that*

$$x \cdot f(t,x) \leqq -\lambda(t)|x|^2 \text{ on } H_0 \times R, \tag{18}$$

where $\lambda(t)$ is a positive continuous function (on H_0) satisfying

$$\int_1^\infty \lambda(t)\,dt = \infty. \tag{19}$$

Then (1) *is of type* (D bis).

Remark. Corresponding to (ii*) *and* to the Remark made after (ii*), the assumption (18) of (ii*) can be relaxed in various directions (cf. also [13], pp. 557-558).

In order to prove (ii* bis), note that (18) implies (13) and so, in view of (ii*), the (B*)-character of (1). This means that, even if (19) is not assumed, all solutions $x(t)$ of (1) exist for $0 \leqq t < \infty$ and lead to a finite limit $x(\infty) \geqq 0$. The assertion of (ii* bis) is that $s(\infty) > 0$ is impossible if (19) is assumed. But it is clear from (1) and (18) that

$$ds(t)/dt \leqq -\lambda(t)s(t), \tag{20}$$

where $s \geqq 0$, $\lambda > 0$ (or just $\lambda \geqq 0$) and $0 \leqq t < \infty$. If (20) is treated in the same way as (17) was treated above, then the assumption $s(\infty) \neq 0$ leads to a contradiction with (19).

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MAXIMAL SUBALGEBRAS OF $C(\Gamma)$.*

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1. Introduction. In [2], Arens and Singer have presented a generalization of part of the theory of analytic functions in the unit disc, established by observing the role played in the classical theory by the group of integers (and the non-negative portion thereof) and replacing this group by a locally compact abelian group G possessing a suitably distingushed semigroup G_+.

We present here a study of the extension to this context of the following result, established by Wermer [3].

THEOREM 1.1. *Let $\mathfrak{A}$ be the Banach algebra of continuous functions on the unit circle, $|z| = 1$, which can be extended to the unit disc $|z| \leqq 1$ so as to be analytic in the interior. The norm is the uniform norm on $|z| = 1$. Then $\mathfrak{A}$ is a maximal subalgebra of the Banach algebra C of all continuous complex-valued functions on the unit circle.*

Specifically, we shall show that if, in the Arens-Singer context, G is a discrete abelian group with a semigroup G_+ satisfying the conditions

$$G_+ \cup G_+^{-1} = G;\ G_+ \text{ is a maximal semigroup in } G, \tag{1.21}$$

the Wermer result is valid if and only if G is archimedean-linearly ordered and G_+ is the set of elements not less than the identity. If the algebra of "analytic" functions is not maximal, we describe completely a maximal subalgebra of C which contains it. A by-product of this investigation is a very brief proof of Theorem 1.1 (see Section 5).

2. Generalized analytic functions. We describe briefly in this section the results of [2] which are pertinent to the present study.

Let G be a discrete abelian group containing a semigroup G_+ such that $G_+ \cup G_+^{-1} = G$. Let A_1 be the Banach algebra of summable functions on G which vanish outside G_+, using the multiplication and the norm defined by

$$(f * g)(x) = \int_{G_+} f(x-y)g(y)\,dy \quad \text{and} \quad \|f\|_1 = \int_{G_+} |f(x)|\,dx.$$

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Let Δ be the set of all multiplicative mappings of the semigroup G_+ into the unit disc of the complex plane. There is a one-one correspondence between the elements ζ in Δ and the homomorphisms h of the Banach algebra A_1 onto the complex numbers, defined by $h(f) = \int_{G_+} f(x)\zeta(x)dx$. Through this correspondence we give to Δ the standard topology of the space of complex homomorphisms (regular maximal ideals) of A_1. This topology is identical with the topology of pointwise convergence of elements of Δ as functions on G_+, and is compact Hausdorff.

Each character of the group G defines a multiplicative functional on G_+, and distinct characters of G define distinct functionals since $G_+ \cup G_+^{-1} = G$. This embedding of characters in Δ is a homeomorphism of the character group Γ with a closed subset of Δ.

The canonical Gelfand representation of the Banach algebra A_1 makes correspond to each f in A_1 a continuous function f^T on Δ defined by $f^T(\zeta) = \int_{G_+} f(x)\zeta(x)dx$. The mapping of f into f^T is an isomorphism of A_1 with an algebra $A_1{}^T$ of continuous functions on Δ.

In the classical situation, when G is the additive group of integers and G_+ is the semigroup of non-negative integers, the space Δ is the unit disc of the complex plane and Γ is the unit circle. The representing functions f^T are those functions continuous on Δ, analytic in the interior, with an absolutely summable sequence of Taylor coefficients.

Each ζ in Δ has a "polar decomposition" $\zeta = \rho\alpha$, where α is in Γ and ρ is a non-negative functional in Δ. The functional ρ is unique; however, α is not generally unique (though it is if ζ has no zeroes on G_+).

The character group Γ is the Silov boundary of the space Δ of maximal ideals of A_1. That is, Γ is the unique minimal closed subset of Δ on which all the representing functions f^T assume their maximum moduli. Thus, in accordance with a general principle of Arens and Singer [1], there corresponds to each point ζ in Δ a regular Baire measure m_ζ on the boundary Γ, such that for any f in A_1,

$$f^T(\zeta) = \int_\Gamma f^T(\alpha) m_\zeta(d\alpha). \tag{2.41}$$

The measure m_ζ generalize (via 2.41) the familiar Poisson integral formula of the classical situation.

A complex-valued function g on Δ is called analytic on Δ if g can be uniformly approximated on Δ by functions f^T with f in A_1. A function is called analytic in the interior of Δ, i. e., $\Delta - \Gamma$, if it can be uniformly approxi-

mated on compact subsets of $\Delta - \Gamma$ by functions f^T, f in A_1. Thus, each f^T is analytic; and, in particular, each element x of the semigroup G_+ defines an analytic function x^T on Δ by

$$x^T(\zeta) = \zeta(x). \tag{2.51}$$

We shall at times wish to think of x^T restricted to Γ, that is, consider the character of Γ which x defines. To avoid confusion, we shall therefore call χ_x the character of Γ which an x in G defines, so that

$$\chi_x(\alpha) = \alpha(x) = \langle x, \alpha \rangle. \tag{2.52}$$

3. Subalgebras. If C is a (commutative) Banach algebra and A is a subalgebra of C, we call A a *maximal subalgebra* of C if $A \neq C$ and if, for any subalgebra B of C for which $A \subset B \subset C$, either $A = B$ or $B = C$.

If A is any subalgebra of a Banach algebra B (with unit), there is a natural continuous mapping of the space $\mathfrak{M}_B$, of complex homomorphisms of B, into the space $\mathfrak{M}_A$. This mapping is defined by simply restricting a homomorphism of B to the subalgebra A. The mapping is in general not one-one. We point out that if h is a homomorphism of B and ϕ is an element of A, we may speak of the vaule of the representing function ϕ^T at the point h as $\phi^T(h)$, without any confusion. For, the value of ϕ^T at h is the same whether ϕ is viewed as an element of A or an element of B.

We are interested here in subalgebras of the Banach algebra $C(\Gamma)$ of all continuous complex-valued functions on a compact Hausdorff space Γ. The norm is the uniform norm. If A is a subalgebra of $C(\Gamma)$, there is a continuous mapping of Γ into the space $\mathfrak{M}_A$. This mapping sends a point α in Γ into the homomorphism h_α defined by $h_\alpha(f) = f(\alpha)$, $f \in A$. The image of Γ under this mapping is a closed subset of $\mathfrak{M}_A$ which obviously includes Ω, the Silov boundary of $\mathfrak{M}_A$.

If h_0 is a complex homomorphism of A, then [1; 7.1], there is a regular Baire measure ν_0 on Ω such that $h_0(f) = \int_\Omega f^T(\omega)\nu_0(d\omega)$. Now ν_0 induces, via the mapping of Γ into $\mathfrak{M}_A$, a measure μ_0 on Γ such that

$$h_0(f) = \int_\Gamma f(\alpha)\mu_0(d\alpha), \qquad f \in A. \tag{3.11}$$

We have used the fact that $f^T(h_\alpha) = f(\alpha)$.

3.2 THEOREM. *Let A be a subalgebra of $C(\Gamma)$ such that the closed linear space spanned by the functions in A and their complex conjugates is*

all of $C(\Gamma)$. Let B be a subalgebra of $C(\Gamma)$ containing A. If a complex homomorphism h_0 of A is given by the measure μ_0 on Γ (3.11), and if h_0 extends to a homomorphism of B, then h_0 extends uniquely to the element h of $\mathfrak{M}_B$ defined by

$$h(f) = \int_\Gamma f(\alpha)\mu_0(d\alpha), \qquad f \in B. \tag{3.21}$$

Consequently, $\mathfrak{M}_B$ is homeomorphic to the closed subset of $\mathfrak{M}_A$ of those homomorphisms h_0 (3.11) for which h (3.21) is a multiplicative functional on B.

Proof. If h_0 extends to a multiplicative functional h on B, then h is of the form $h(f) = \int_\Gamma f(\alpha)\mu(d\alpha)$, $f \in B$ for some measure μ on Γ. Since the functions in A, together with their conjugates, span $C(\Gamma)$, a positive functional on $C(\Gamma)$ is completely determined by its effect on the functions in A. Since μ, as a functional on $C(\Gamma)$, agrees with μ_0 on A, we must have $\mu = \mu_0$, i.e., h must be of the form (3.21).

Thus, the natural continuous mapping of $\mathfrak{M}_B$ into $\mathfrak{M}_A$ is one-one, and because $\mathfrak{M}_B$ is compact Hausdorff, the mapping is a homeomorphism of $\mathfrak{M}_B$ with its image in $\mathfrak{M}_A$.

4. The maximality theorem. Returning to the setting of Section 2, let $\mathfrak{A}$ be the Banach algebra of continuous functions on the boundary Γ which can be extended analytically to the interior of Δ. That is, $\mathfrak{A}$ is the completion under the uniform norm of the algebra of functions obtained by restricting functions in A_1^T to the character group Γ.

4.1 THEOREM. *For the continuous function f on Γ to belong to $\mathfrak{A}$, it is necessary and sufficient that*

$$\int_\Gamma \overline{\langle x, \alpha\rangle} g(\alpha)\, d\alpha = 0 \tag{4.11}$$

for each x in G which is not in G_+.

Proof. The necessity is evident from the fact that the property is possessed by functions in A_1^T. Suppose f has the stated property. For each $\epsilon > 0$ there is a continuous function ϕ on Γ such that $\| f - f * \phi \|_\infty < \epsilon$. By the Plancherel theorem, the Fourier transform of $f * \phi$ is a function F in $L_1(G)$, and the Fourier transform of F is $f * \phi$. But since the Fourier transform of f vanishes off G_+, we see that F is in $L_1(G_+) = A_1$. Thus f can be uniformly approximated on Γ by functions F^T with F in A_1. Consequently, f is in $\mathfrak{A}$.

We now ask when $\mathfrak{A}$ is a maximal subalgebra of $C(\Gamma)$. Clearly, it is necessary that G_+ be a maximal semigroup in G, for any larger (proper) semigroup would immediately yield an algebra properly between $\mathfrak{A}$ and $C(\Gamma)$, e.g., the functions in $C(\Gamma)$ "analytic" with respect to this larger semigroup. However, the maximality of G_+ is not sufficient for the maximality of $\mathfrak{A}$, as we shall soon show.

First, let us make a few observations under the assumptions (1.21). Let

$$G_+ = G_0 \cup G_1, \tag{4.21}$$

where G_0 is the subgroup of G consisting of the elements of G_+ having their inverses in G_+. We distinguish the functional ρ_0 in Δ defined by

$$\rho_0(x) = 1, \quad x \in G_0, \qquad \rho_0 = 0, \quad x \in G_1. \tag{4.22}$$

If we consider the quotient group

$$\mathcal{G} = G/G_0, \tag{4.23}$$

it is easy to see that

$$\mathcal{G}_+ = G_+/G_0 \tag{4.24}$$

is a sub-semigroup of this group. The character group of $\mathcal{G}$ is (of course) isomorphic and homeomorphic to the closed subgroup Λ of Γ which consists of the characters of G which are identically 1 on G_0.

We shall now assemble some useful facts concerning $\mathcal{G}$, $\mathcal{G}_+$, ρ_0, etc.

4.3 LEMMA. *Under the conditions* (1.21),

i) *the semigroup $\mathcal{G}_+$ defines an archimedean-linear order on the group $\mathcal{G}$*

ii) *if x is in G_1* (4.21), *the analytic function x^γ* (2.51) *has zeros on Δ exactly at the points $\zeta = \rho_0\alpha$, α in Γ*

iii) *the measure μ_0 on Γ associated with ρ_0 by* (2.41) *is the Haar measure of the subgroup Λ,* i.e., *$\mu_0(E) = \mu(E \cap \Lambda)$ for a Borel set E in Γ, where μ is the Haar measure of the compact group Λ.*

Proof. i) From the definition of G_0, we see that the semigroup $\mathcal{G}_+$ has only the identity element of $\mathcal{G}$ in common with $\mathcal{G}_+^{-1}$. Consequently, $\mathcal{G}$ is totally (linearly) ordered by the semigroup $\mathcal{G}_+$. Since G_+ is a maximal semigroup in G, $\mathcal{G}_+$ is a maximal semigroup in $\mathcal{G}$. Let ξ and η be elements of $\mathcal{G}_+$, both different from the identity of $\mathcal{G}$. Then there must be a positive integer n such that $\xi^n > \eta$, i.e., such that $\xi^n\eta^{-1}$ is in $\mathcal{G}_+$. For, if this is not the case, the set of elements $\xi^{-n}\theta$, θ in $\mathcal{G}_+$, is a semigroup properly larger than $\mathcal{G}_+$ (since ξ^{-1} has been adjoined) and is not all of $\mathcal{G}$, since $\xi^{-n}\theta > \eta^{-1}$

for every θ in $\mathcal{G}_+$. Hence, $\mathcal{G}_+$ defines an archimedean-linear order on $\mathcal{G}$, with $\mathcal{G}_+$ the set of elements not less than the identity. ii) If x is in G_1, then $x^T(\rho_0\alpha) = \rho_0(x)\alpha(x) = 0 \cdot \alpha(x) = 0$. Conversely, suppose that $x^T(\rho\alpha) = 0$. Since G_0 is a group, ρ must be identically 1 on G_0. Hence, ρ defines a multiplicative mapping ρ_1 of $\mathcal{G}_+$ into the unit interval. If η is the coset of x (modulo G_0), then $\rho_1(\eta) = \rho(x) = 0$. If ξ is any element of $\mathcal{G}_+$ other than the identity, there is a positive integer n such that $\xi^n\eta^{-1}$ is in $\mathcal{G}_+$. Then

$$\rho_1(\xi)^n = \rho_1(\xi^n) = \rho_1(\eta)\rho_1(\eta^{-1}\xi^n) = 0.$$

Thus ρ_1 is the functional which is 1 at the identity of $\mathcal{G}$ and is 0 at all other elements of $\mathcal{G}_+$. Consequently, ρ must be 1 on G_0 and 0 on the remainder of G_+, i.e., $\rho = \rho_0$. iii) The measure μ_0 has the Fourier-Stieltjes transform

$$\mu_0{}^T(x) = \int_\Gamma \overline{\langle x, \alpha\rangle}\mu_0(d\alpha) = 1 \text{ if } x \in G_0, \text{ and } = 0 \text{ if } x \in G - G_0.$$

On the other hand, if μ is Haar measure on Λ,

$$\int_\Lambda \overline{\langle x, \alpha\rangle}\mu(d\alpha) = 1 \text{ if } x \in G_0, \text{ and } = 0 \text{ if } x \in G - G_0.$$

Since a measure on Γ is completely determined by its Fourier-Stieltjes coefficients, we must have $\mu_0(E) = \mu(E \cap \Lambda)$ for each Borel set E.

We now define our candidate for a maximal subalgebra of $C(\Gamma)$ containing $\mathfrak{A}$. Let $\mathfrak{A}_0$ be the subalgebra of $C(\Gamma)$ of those functions f for which

$$\int_\Gamma \overline{\langle x, \alpha\rangle}f(\alpha)\mu_0(d\alpha) = 0, \qquad x \notin G_+.$$

In other words, $\mathfrak{A}_0$ is the collection of continuous functions on Γ whose restrictions to Λ have Fourier transforms that vanish outside the semigroup $\mathcal{G}_+$ (4.24). When $\mathfrak{A}_0$ is described in this way, one can see easily that $\mathfrak{A}_0$ is an algebra, as we have asserted.

4.5 LEMMA. *$\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_0$.*

Proof. Suppose f is in $\mathfrak{A}$. Then for any x in $G - G_+$,

$$\int \overline{\langle x, \alpha\rangle}f(\alpha)\mu_0(d\alpha) = \int_\Gamma \langle x^{-1}, \alpha\rangle f(\alpha)\mu_0(d\alpha). \tag{4.51}$$

The first condition of (1.21) tells us that x^{-1} is in G_+. Then, by (2.41) the right integral of (4.51) is

$$(x^{-1})^T(\rho_0)\int_\Gamma f(\alpha)\mu_0(d\alpha) = \rho_0(x^{-1})\int_\Gamma f(\alpha)\mu_0(d\alpha) = 0$$

Thus f belongs to $\mathfrak{A}_0$.

4.6 THEOREM. *The space $\mathfrak{S}_{\mathfrak{A}_0}$ of complex homomorphisms of $\mathfrak{A}_0$ is the set of points ζ in Δ for which either* 1) $\zeta(x) = 1$ *for all x in G_0 or* 2) ζ *is in* Γ.

Proof. Since $\mathfrak{A} \subset \mathfrak{A}_0$, we know from (3.2) that $\mathfrak{S}_{\mathfrak{A}_0}$ is homeomorphic to the closed subset of Δ containing the points ζ for which (see (2.41))

$$h(f) = \int_\Gamma f(\alpha)m_\zeta(d\alpha) \tag{4.61}$$

defines a multiplicative functional h on $\mathfrak{A}_0$. Let C_0 denote the algebra of continuous functions on Γ which vanish on Λ. Its space of maximal ideals is $\Gamma - \Lambda$. Furthermore, $C_0 \subset \mathfrak{A}_0$. Suppose h (4.61) is multiplicative on $\mathfrak{A}_0$, and let h_0 denote the restriction of h to C_0.

If $h_0 \equiv 0$, then the carrier of the measure m_ζ lies in Λ. If $x \in G_0$ and $\alpha \in \Lambda$, then $\langle x, \alpha\rangle = 1$, so that

$$1 = \int_\Gamma \langle x, \alpha\rangle m_\zeta(d\alpha) = x^T(\zeta) = \zeta(x).$$

Thus 1) holds.

If $h_0 \not\equiv 0$, then there exists a point γ in $\Gamma - \Lambda$ such that for every f in C_0

$$\int_\Gamma f(\alpha)m_\zeta(d\alpha) = h(f) = h_0(f) = f(\gamma).$$

Let N be any open set containing γ such that $N \cap \Lambda$ is empty. Let g_N be a non-negative continuous function on Γ which is identically one on $\Gamma - N$ and vanishes at γ. Then $1 - g_N$ belongs to C_0 and

$$\int_\Gamma (1 - g_N(\alpha))m_\zeta(d\alpha) = 1 - g_N(\gamma) = 1.$$

Since $m_\zeta(\Gamma) = 1$, we have $\int_\Gamma g_N(\alpha)m_\zeta(d\alpha) = 0$. Therefore the measure m_ζ is concentrated at the point γ. Thus $\zeta = \gamma$ and 2) holds.

If 2) holds, h (4.61) is obviously multiplicative on $\mathfrak{A}_0$. If 1) holds,

ζ defines a multiplicative functional on $\mathcal{G}_+$ (4.24) and hence a homomorphism of $\mathfrak{A}_0/C_0$, thus a homomorphism of $\mathfrak{A}_0$.

Our maximality theorem is as follows.

4.7 THEOREM. *The algebra $\mathfrak{A}_0$ is a maximal subalgebra of $C(\Gamma)$ containing $\mathfrak{A}$. In order that $\mathfrak{A}$ be maximal, i.e. that $\mathfrak{A} = \mathfrak{A}_0$, it is necessary and sufficient that G be archimedean-linearly ordered, with G_+ the semigroup of elements not less than the identity.*

Proof. To see that $\mathfrak{A}_0$ is maximal. we assume that $\mathfrak{B}$ is a subalgebra of $C(\Gamma)$ which contains $\mathfrak{A}_0$. We know from (4.6) that ρ_0 (4.22) defines a homomorphism of $\mathfrak{A}_0$. (i) If the ρ_0 homomorphism extends to $\mathfrak{B}$, it does so in the form

$$h(f) = \int_\Gamma f(\alpha)\mu_0(d\alpha). \tag{4.71}$$

Let f be in $\mathfrak{B}$. Then for any x in $G - G_+$,

$$\int_\Gamma \overline{\langle x, \alpha\rangle} f(\alpha)\mu_0(d\alpha) = \int_\Gamma \langle x^{-1}, \alpha\rangle f(\alpha)\mu_0(d\alpha) = \rho_0(x^{-1}) \int_\Gamma f(\alpha)\mu_0(d\alpha).$$

But x^{-1} is an element of G_+ not in G_0, so that $\rho_0(x^{-1}) = 0$. Consequently, for x in $G - G_+$,

$$\int_\Gamma \overline{\langle x, \alpha\rangle} f(\alpha)\mu_0(d\alpha) = 0.$$

Thus, f is in $\mathfrak{A}_0$. Therefore $\mathfrak{B} = \mathfrak{A}_0$. (ii) If the ρ_0 homomorphism (4.71) does not extend to $\mathfrak{B}$, we shall show that $\mathfrak{B} = C(\Gamma)$. Let x be an element of $G_1 = G_+ - G_0$. From part (ii) of (4.3) we know that the only zeros of x^T on Δ are at the points $\zeta = \rho_0\alpha$, α in Γ. On the other hand, (4.6) tells us that the only such point in $\mathcal{S}_{\mathfrak{A}_0}$ is ρ_0. Consequently, viewing χ_x (2.52) as an element of $\mathfrak{B}$, we see that its representing function has no zeros on $\mathcal{S}_{\mathfrak{B}}$ (ρ_0 having been removed). Therefore $\chi_x^{-1} = \chi_{x^{-1}}$ is in $\mathfrak{B}$. Thus $\mathfrak{B}$ contains all characters of Γ and must be all of $C(\Gamma)$.

The condition that $\mathfrak{A} = \mathfrak{A}_0$ is evidently that the subgroup Λ be all of Γ; for, if this is not the case, $\mathfrak{A}$ must be properly included in $\mathfrak{A}_0$ because $\mathfrak{A}_0$ contains all functions in $C(\Gamma)$ which vanish on Λ. But, the condition that $\Lambda = \Gamma$ is simply the condition that G_0 reduces to the identity, i.e., that G is isomorphic to the archimedean-linearly ordered group $\mathcal{G}$ (see 4.3, part (i)).

An interesting example of the situation which this theorem describes is obtained by letting G be the additive group of the Gaussian integers, that

is, the direct sum of the group of integers with itself. The most natural semigroup to consider is the pairs of integers (m, n) for which $m \geqq 0$, $n \geqq 0$. However, this semigroup does not satisfy $G_+ \cup G_+^{-1} = G$. This can be remedied by adding to the set the pairs (m, n) with $m < 0$ and $n > 0$. Now we have a semigroup satisfying the first condition of (1.21). However, this is not a maximal semigroup, so we extend it to the set G_+ of pairs (m, n) for which $n \geqq 0$, m arbitrary. Then G_+ is a maximal semigroup. The character group Γ is then the torus; the subgroup G_0 of G is the set of pairs $(m, 0)$; the subgroup Λ of Γ is a unit circle on the torus, in fact, $\Gamma = \Lambda \times \Lambda$. The algebra $\mathfrak{A}$ consists of the continuous functions f on the torus such that, for every integer m, the function g_m on the unit circle defined by $g_m(\theta) = \int_0^{2\pi} f(\psi, \theta) e^{im\psi}\, d\psi$ has a Fourier transform vanishing on the negative integers. The maximal subalgebra $\mathfrak{A}_0$ is the collection of continuous functions f for which the function $g(\theta) = f(0, \theta)$ is an "analytic" function on the unit circle.

5. The classical case. It would seem worthwhile at this point to see how these methods apply to the classical situation, in order that the simplicity of the argument in that case should not be lost in the complexities of the more general situation. The argument which we present applies with virtually no change to all cases where G is archimedean-linearly ordered (subgroup of the additive group of real numbers) and G_+ is the semigroup of non-negative elements.

Proof of Theorem 1.1. Suppose that $\mathfrak{B}$ is a subalgebra of C which contains $\mathfrak{A}$. The origin, $z = 0$, defines a homomorphism h of $\mathfrak{A}$ by

$$h(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta. \tag{5.11}$$

(i) Suppose that the homomorphism (5.11) extends to $\mathfrak{B}$. This homomorphism is represented by a measure μ on $|z| = 1$, and since any measure on the unit circle is completely determined by the Fourier-Stieltjes coefficients $\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \mu(d\theta)$, $n = 0, 1, 2, \cdots$, the measure μ must be (normalized) Lebesgue measure. Consequently, for any f in $\mathfrak{B}$ and any positive integer n,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = z^n(0) \cdot \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta = 0.$$

Thus, the Fourier transform of f vanishes on the negative integers, i.e., f belongs to $\mathfrak{A}$. (ii) If the homomorphism (5.11) does not extend to $\mathfrak{B}$,

the functions $\chi_n(\theta) = e^{in\theta}$, $n = 1, 2, \cdots$ belong to no maximal ideal of $\mathfrak{B}$, as the analytic functions z^n vanish only at $z = 0$. Consequently, each χ_n is invertible in $\mathfrak{B}$. Therefore, $\mathfrak{B}$ contains all characters χ_n, $n = 0, \pm 1, \pm 2, \cdots$, so that $\mathfrak{B} = C$.

6. Concluding remarks. The method of this paper is a special case of a general procedure for finding maximal subalgebras $\mathfrak{A}_0$ of $C(X)$ which separate the points of X. If $\mathfrak{A}$ is a "separating" subalgebra of $C(X)$, then its space of maximal ideals $\mathcal{M}_{\mathfrak{A}}$ contains X. Since the norm of a function in $\mathfrak{A}$ is its sup over X, X contains the Silov boundary Γ of $\mathfrak{A}$. Let Λ be a closed subset of X and let $\mathfrak{A}_\Lambda$ denote the closure of the restriction of functions in $\mathfrak{A}$ to the set Λ, so that $\mathfrak{A}_\Lambda$ is a subalgebra of $C(\Lambda)$. Let $\mathfrak{A}_0$ be the subalgebra of $C(X)$ consisting of all continuous functions on X whose restriction to Λ lies in $\mathfrak{A}_\Lambda$. $\mathfrak{A} \subset \mathfrak{A}_0$; and if $\mathfrak{A}_\Lambda \neq C(\Lambda)$, then $\mathfrak{A}_0 \neq C(X)$. Thus $\mathfrak{A}$ cannot be a maximal subalgebra if $\mathfrak{A}_\Lambda \neq C(\Lambda)$ and $\mathfrak{A} \neq \mathfrak{A}_0$. This is the situation in the non-archimedean ordered case of Section 4, where we go on to prove that $\mathfrak{A}_0$ is maximal because $\mathfrak{A}_\Lambda$ is maximal in $C(\Lambda)$. Thus the maximal subalgebras $\mathfrak{A}$ of $C(X)$ are of two kinds. Of the first type are those for which there exists a closed subset Λ such that $\mathfrak{A}_\Lambda$ is maximal in $C(\Lambda)$ and $\mathfrak{A} = \mathfrak{A}_0$ [though it is not clear that all such algebras are maximal]; of the second type are those for which no such subset exists. For the first type, the problem "reduces" to finding the maximal subalgebras of $C(\Lambda)$.

Note further that in order for $\mathfrak{A}$ to be maximal in $C(X)$, the Silov boundary $\Gamma = X$. For the map: $f \to f|_\Gamma$ is norm preserving and therefore gives an isomorphism of $\mathfrak{A}$ with $\mathfrak{A}_\Gamma$. If $\mathfrak{A}_\Gamma = C(\Gamma)$, then $\Gamma = \mathcal{M}_{\mathfrak{A}_\Gamma} = \mathcal{M}_{\mathfrak{A}} \supset X$, so that $\Gamma = X$. If $\mathfrak{A}_\Gamma \neq C((\Gamma)$, then $\mathfrak{A}_0 = \mathfrak{A}$ (by maximality of $\mathfrak{A}$) so that every function in $\mathfrak{A}_0$ has a unique extension to X. In particular, the zero function on Γ has a unique extension to X, i.e., $\Gamma = X$.

So in seeking maximal subalgebras we can assume the Silov boundary is X. Suppose $\mathcal{M}_{\mathfrak{A}} \neq X$. Consider the class $\mathcal{C}$ of subalgebras of $C(X)$ with the same space of maximal ideals $\mathcal{M}_{\mathfrak{A}}$. Let $\mathfrak{A}_\alpha$ be an increasing family (under inclusion) of algebras in $\mathcal{C}$. Let $\mathfrak{B}$ be the closure of the algebra $\bigcup_\alpha \mathfrak{A}_\alpha$. One can easily check that $\mathfrak{B} \in \mathcal{C}$. Consequently, Zorn's lemma applies and we conclude that every $\mathfrak{A} \in \mathcal{C}$ is contained in a maximal $\mathfrak{B} \in \mathcal{C}$. Since $\mathcal{M}_{\mathfrak{B}} = \mathcal{M}_{\mathfrak{A}} \neq X$, $\mathfrak{B} \neq C(X)$. Therefore, in order for $\mathfrak{A}$ to be maximal in $C(X)$, then $\mathfrak{A}$ must be maximal in $\mathcal{C}$.

Suppose, then, that $\mathfrak{A}$ is maximal in $\mathcal{C}$ and $\mathfrak{A} \subsetneq \mathfrak{B} \subset C(X)$. Then $\mathcal{M}_{\mathfrak{B}} \neq \mathcal{M}_{\mathfrak{A}}$ and by Theorem 3.2 (under the conditions of this theorem)

$\mathcal{S}_{\mathfrak{B}} \subsetneq \mathcal{S}_{\mathfrak{A}}$. The problem now is to show that there are sufficiently many functions $\{f_\alpha\}$ in $\mathfrak{A}$ whose zeros lie entirely in $\mathcal{S}_{\mathfrak{A}} - \mathcal{S}_{\mathfrak{B}}$ $(f_\alpha^{-1} \in \mathfrak{B})$, such that $\mathfrak{A}$, $\{f_\alpha\}$ generate all of $C(X)$, i.e., $\mathfrak{B} = C(X)$. The proof of the main theorem is the verification that $\mathfrak{A}$ is maximal in $\mathcal{C}$ and enough such functions f_α exist.

It was pointed out to us by Wermer and de Leeuw that the sup norm is not essential to these arguments.

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ON THE TOTAL CURVATURE OF IMMERSED MANIFOLDS.*

By Shiing-shen Chern[1] and Richard K. Lashof.

Introduction. In the classical theory of surfaces in the ordinary Euclidean space E an important rôle is played by the normal mapping of Gauss: Let M be an oriented surface which has at every point x a well-defined unit normal vector $\nu(x)$. Then the normal mapping $\nu: M \to S_0$ is the mapping of M into the unit sphere S_0 about the origin of E, which sends x to $\nu(x)$.

For differentiably immersed submanifolds in an Euclidean space of higher dimension the following is a generalization of the Gauss mapping: We consider a C^∞-manifold M^n of dimension n, and a C^∞-mapping $x: M^n \to E^{n+N}$ into the Euclidean spase E^{n+N} of dimension $n+N$ $(N \geqq 1)$. M^n, or rather M^n and the mapping x, is called an immersed submanifold, if the induced mapping of the tangent space is univalent everywhere, or, what is the same, if the Jacobian matrix of x is everywhere of rank n. The submanifold M^n is said to be imbedded, if x is one-one; that is, if $x(p) = x(q)$, $p, q \in M^n$, implies that $p = q$. Let B_ν be the bundle of unit normal vectors of $x(M^n)$, so that a point of B_ν is a pair $(p, \nu(p))$, where $\nu(p)$ is a unit normal vector to $x(M^n)$ at $x(p)$. Then B_ν is a bundle of $(N-1)$ dimensional spheres over M^n and is a C^∞-manifold of dimension $n+N-1$. The mapping $\nu: B_\nu \to S_0^{n+N-1}$ of B_ν into the unit sphere S_0^{n+N-1} of E^{n+N} defined by $\tilde{\nu}(p, \nu(p)) = \nu(p)$ is the mapping with which we will be concerned in this paper.

Let dV be the volume element of M^n. There is a differential form $d\sigma_{N-1}$ of degree $N-1$ on B_ν such that its restriction to a fiber is the volume element of the sphere of unit normal vectors at a point $p \in M^n$; then $d\sigma_{N-1} \wedge dV$ is the volume element of B_ν (for detail see Section 2). Let $d\Sigma_{n+N-1}$ be the volume element of S_0^{n+N-1}. The function $G(p, \nu(p)) = G(p, \nu)$ defined by

$$\tilde{\nu}^* d\Sigma_{n+N-1} = G(p, \nu)\, d\sigma_{N-1} \wedge dV, \tag{1}$$

where $\tilde{\nu}^*$ is the dual mapping on differential forms induced by $\tilde{\nu}$, is a function in B_ν. It generalizes the Gauss-Kronecker curvature and we will call

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it the Lipschitz-Killing curvature at $\nu(p)$. $G(p,\nu)$ has a geometrical interpretation which we will discuss below. It is zero at a point $(p,\nu(p)) \in B_\nu$, if and only if $\tilde{\nu}$ has a critical value at this point. We call the integral

$$K^*(p) = \int |G(p,\nu)|\, d\sigma_{N-1} \geqq 0 \tag{2}$$

over the sphere of unit normal vectors at $x(p)$ the *total curvature of* M^n *at* p, and define as the *total curvature of* M^n itself the integral $\int_{M^n} K^*(p)dV$, if it converges.

The main results of this paper are concerned with the conclusions on M^n when its total curvature is "small." They can be stated as the following theorems:

THEOREM 1. *Let* M^n *be a compact oriented* C^∞*-manifold immersed in* E^{n+N}. *Its total curvature satisfies the inequality:*[2]

$$\int_{M^n} K^*(p)dV \geqq 2c_{n+N-1} \tag{3}$$

THEOREM 2. *Under the hypothesis of Theorem* 1, *if*

$$\int_{M^n} K^*(p)dV < 3c_{n+N-1}, \tag{4}$$

then M^n *is homeomorphic to a sphere of* n*-dimensions.*

THEOREM 3. *Under the same hypothesis, if*

$$\int_{M^n} K^*(p)dV = 2c_{n+N-1}, \tag{5}$$

then M^n *belongs to a linear subvariety* E^{n+1} *of dimension* $n+1$, *and is imbedded as a convex hypersurface in* E^{n+1}. *The converse of this is also true.*

These theorems generalize known results of Fenchel, Fary [2], and Milnor [3] for curves and sharpen some results of Milnor and one of us [1]. Theorem 3 can be interpreted as a characterization of convex hypersurfaces among all immersed submanifolds of a given dimension in an Euclidean space of arbitrary dimension. A large part of our paper is devoted to a proof of this theorem.

1. Moving frames. Suppose E^{n+N} be oriented. By a frame $xe_1 \cdots e_{n+N}$ in E^{n+N} we mean a point x and an ordered set of mutually perpendicular

[2] c_{n+N-1} is the area of the unit hypersphere in an Euclidean space of dimension $n+N$.

unit vectors $e_1, \cdots, e_{n+N}$, such that their orientation is coherent with that of E^{n+N}. Unless otherwise stated, we agree on the following ranges of the indices:

$$1 \leqq i, j, k \leqq n, \quad n+1 \leqq r, s, t \leqq n+N, \quad 1 \leqq A, B, C \leqq n+N. \tag{6}$$

Then we have

$$e_A e_B = \delta_{AB}, \tag{7}$$

where the left-hand side is the scalar product of vectors. Let $F(n, N)$ be the space of all frames in E^{n+N}, so that $\dim F(n, N) = (n+N)(n+N+1)/2$. In $F(n, N)$ we introduce the linear differential forms ω'_A, ω'_{AB} by the equations

$$de_A = \sum_B \omega'_{AB} e_B, \qquad dx = \sum_A \omega'_A e_A, \tag{8}$$

where

$$\omega'_{AB} + \omega'_{BA} = 0. \tag{9}$$

Their exterior derivatives satisfy the equation of structure:

$$d\omega'_A = \sum_B \omega'_B \wedge \omega'_{BA}, \qquad d\omega'_{AB} = \sum_C \omega'_{AC} \wedge \omega'_{CB}. \tag{10}$$

As explained in the Introduction, we mean by an immersed submanifold in E^{n+N} an abstract C^∞-manifold M^n and a C^∞-mapping $x: M^n \to E^{n+N}$, such that the induced mapping x_* on the tangent space is everywhere univalent. Analytically, the mapping can be defined by a vector-valued function $x(p)$, $p \in M^n$. Our assumption implies that the differential $dx(p)$ of $x(p)$, which is a linear differential form in M^n, with value in E^{n+N}, has as values a linear combination of n vectors, $t_1, \cdots, t_n$, and not less. Since x_* is univalent, we can identify the tangent space of M^n at p with the vector space spanned by $t_1, \cdots, t_n$. A linear combination of the latter is called a tangent vector and a vector perpendicular to them is called a normal vector. The immersion of M^n in E^{n+N} gives rise to the following fiber bundles over M^n:

1) The tangent bundle B_T, whose bundle space is the subset of $M^n \times E^{n+N}$, consisting of all points (p, v), such that $p \in M$ and v is a unit tangent vector at $x(p)$.

2) The normal bundle B_ν, whose bundle space is the subset of $M^n \times E^{n+N}$, consisting of all points (p, v), such that $p \in M$ and v is a unit normal vector at $x(p)$.

3) The bundle B, whose bundle space is the subset of $M^n \times F(n, N)$, consisting of $(p, x(p)e_1 \cdots e_n e_{n+1} \cdots e_{n+N}) \in M^n \times F(n, N)$ such that $e_1, \cdots, e_n$

are tangent vectors and $e_{n+1}, \cdots, e_{n+N}$ are normal vectors at $x(p)$. The projection $B \to M^n$ we denote by ψ.

The last bundle B is the space in which most of our computations take place.[3] We define the mappings

$$\psi_T : B \to B_T, \qquad \psi_\nu : B \to B_\nu, \tag{11}$$

by

$$\psi_T(p, x(p)e_1 \cdots e_{n+N}) = (p, e_n), \qquad \psi_\nu(p, x(p)e_1 \cdots e_{n+N}) = (p, e_{n+N}). \tag{12}$$

We also remark that the Whitney sum $B_T \oplus B_\nu$ over M^n is equivalent to the product bundle $M^n \times S^{n+N-1} \to M^n$.

Consider the mappings

$$B \xrightarrow{i} M^n \times F(n, N) \xrightarrow{\lambda} F(n, N), \tag{13}$$

where i is inclusion and λ is the projection into the second factor. Put

$$\omega_A = (\lambda i)^* \omega'_A, \qquad \omega_{AB} = (\lambda i)^* \omega'_{AB}, \tag{14}$$

Then we have, from (9) and (10),

$$\omega_{AB} + \omega_{BA} = 0, \tag{15}$$

$$d\omega_A = \sum_B \omega_B \wedge \omega_{BA}, \qquad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \tag{16}$$

From our definition of B it follows that $\omega_r = 0$ and that ω_i are linearly independent. Hence the first equation of (16) gives

$$\sum_i \omega_i \wedge \omega_{ir} = 0.$$

From this it follows that

$$\omega_{ir} = \sum_j A_{rij} \omega_j, \qquad A_{rij} = A_{rji}. \tag{17}$$

2. The total curvature. An essential idea of the method of moving frames is to consider the bundle space B, and to construct differential forms

[3] The bundle B can also be defined as follows: Let Q_{n+N} be the group of all orientation-preserving motions in E^{n+N}, and $R_n \times R_N$ the subgroup of Q_{n+N} consisting of all motions which leave fixed the origin 0 of E^{n+N} and a given oriented linear space of dimension n through 0. Then $Q_{n+N} \to Q_{n+N}/R_n \times R_N$ is a bundle whose structure group is $R_n \times R_N$ and whose base space is the space of all elements consisting of a point and an oriented linear space of dimension n through it. The bundle B is induced by the mapping which sends $p \in M$ to the point $x(p)$ and the oriented tangent space at $x(p)$. In particular, the structural group of B is $R_n \times R_N$. Similarly, we can define the bundles B_T and B_ν.

in B, which are inverse images of differential forms in M^n and B_ν, under the mappings ψ and ψ_ν respectively. In particular, the volume element of M^n can be written

$$dV = \omega_1 \wedge \cdots \wedge \omega_n, \tag{18}$$

and the volume element of B_ν is

$$dV \wedge d\sigma_{N-1} = \omega_1 \wedge \cdots \wedge \omega_{n+N,n+1} \wedge \cdots \wedge \omega_{n+N,n+N-1}, \tag{19}$$

$d\sigma_{N-1}$ being equal to the product of the last $N-1$ factors. On the other hand, we have

$$\tilde{\nu}^* d\Sigma_{n+N-1} = \omega_{n+N,1} \wedge \cdots \wedge \omega_{n+N,n} \wedge \omega_{n+N,n+1} \wedge \cdots \wedge \omega_{n+N,n+N-1}.$$

Using (17), we get

$$\tilde{\nu}^* d\Sigma_{n+N-1} = (-1)^n \det(A_{n+N,ij})\omega_1 \wedge \cdots \wedge \omega_n \wedge \omega_{n+N,n+1} \wedge \cdots \wedge \omega_{n+N,n+N-1}.$$

It follows from the definition (1) of the Lipschitz-Killing curvature $G(p,\nu)$ that

$$G(p,\nu) = (-1)^n \det(A_{n+N,ij}). \tag{20}$$

To see how $G(p,\nu)$ depends on ν, we take a local cross-section of M^n in B, described by the functions $\tilde{e}_A(q)$ for q in a neighborhood of p. Then for any frame $e_A(q)$ in B at $x(q)$, we have $e_A = \sum c_{AB}\tilde{e}_B(q)$ and

$$\sum_{i,j} A_{sij}\omega_i\omega_j = \sum_{r,i,j} c_{sr}\tilde{A}_{rij}\tilde{\omega}_i\tilde{\omega}_j,$$

where $\tilde{A}_{rij}$ is the function A_{rij} restricted to the local cross-section. In particular, if for $\nu = e_{n+N}$ we write $\nu = \sum \nu_r \tilde{e}_r$ we have at p:

$$G(p,\nu) = (-1)^n \det\Big(\sum_r \nu_r \tilde{A}_{rij}(p)\Big). \tag{21}$$

Further we get for the scalar product of ν and the vector-valued second differential d^2x on B

$$\nu \cdot d^2x = \sum_{i,r} \nu_r \tilde{\omega}_{ir}\tilde{\omega}_i = \sum_{r,i,j} \nu_r \tilde{A}_{rij}\tilde{\omega}_i\tilde{\omega}_j \tag{22}$$

and hence as forms on B_ν we have:

$$-d\nu dx = \nu \cdot d^2x = \sum_{r,i,j} \nu_r \tilde{A}_{rij}\tilde{\omega}_i\tilde{\omega}_j. \tag{23}$$

Therefore we may interpret $G(p,\nu)$ as generalizing the determinant of the second fundamental form of a surface.

If $N = 1$, i.e., if M^n is an immersed oriented hypersurface in E^{n+1}, its

orientation (and that of E^{n+1}) defines a unit normal vector $\nu_0(p)$ at $p \in M^n$. Then $G(p, \nu_0(p)) = G(p)$ is called the Gauss-Kronecker curvature of M at p. Any other unit normal vector at p is of the form $\nu(p) = \pm \nu_0(p)$, and

$$G(p, \nu(p)) = G(p, \pm \nu_0(p)) = (\pm 1)^n G(p).$$

It follows that, for n even, $G(p, \nu(p))$ is independent of the orientation of the hypersurface M^n and of the space E^{n+1}. Naturally, $G(p)$ reduces for $n = 2$ to the classical Gaussian curvature.

In the general case $G(p, \nu)$ admits the following geometrical interpretation in terms of the Gauss-Kronecker curvature: Let $L(\nu)$ be the linear space of dimension $n+1$ spanned by the tangent space to $x(M^n)$ at $x(p)$ and the normal vector $\nu(p)$. Then $G(p, \nu)$ is equal to the Gauss-Kronecker curvature at p of the orthogonal projection of $x(M^n)$ into $L(\nu)$.

Since the theorem is local, we take a local cross-section $\tilde{e}_A(q)$ of M^n in B in a neighborhood of p, such that $\nu(p) = \tilde{e}_{n+N}(p)$. We write

$$\tilde{e}_A(p) = (\tilde{e}_A)_0, \qquad x(p) = x_0.$$

If $x'(q)$ denotes the position vector of the projection of $x(q)$ in $L(\nu)$, $x'(q)$ is defined by the equations

$$x'(q) - x(q) = \xi_{n+1}(\tilde{e}_{n+1})_0 + \cdots + \xi_{n+N-1}(\tilde{e}_{n+N-1})_0,$$

$$x'(q) - x_0 \equiv 0, \qquad \mod (\tilde{e}_1)_0, \cdots, (\tilde{e}_n)_0, (\tilde{e}_{n+N})_0.$$

From this it follows that

$$\xi_{n+\lambda} = (x'(q) - x(q)) \cdot (\tilde{e}_{n+\lambda})_0 = (x_0 - x(q)) \cdot (\tilde{e}_{n+\lambda})_0 \quad 1 \leqq \lambda \leqq N-1.$$

If p is fixed and q varies on the manifold M^n, we have

$$dx' = dx + \sum_\lambda d\xi_{n+\lambda}(\tilde{e}_{n+\lambda})_0 = dx - \sum_\lambda (dx \cdot (\tilde{e}_{n+\lambda})_0)(\tilde{e}_{n+\lambda})_0,$$

$$d^2x' = d^2x - \sum_\lambda (d^2x \cdot (\tilde{e}_{n+\lambda})_0)(\tilde{e}_{n+\lambda})_0,$$

so that $(\tilde{e}_{n+N})_0 d^2x' = (\tilde{e}_{n+N})_0 d^2x$. This proves our statement.

3. Proof of Theorem 1. For this proof we will need the following:

THEOREM (Sard)[4] *Let V and W be two C^1-manifolds of the same dimension and f a mapping of class C^1 of V into W. The image $f(E)$ of the set E of critical points of f is a set of measure zero in W.*

[4] Cf. [6], p. 10.

We consider the map $\tilde{\nu}: B_\nu \to S_0^{n+N-1}$. Every point of S_0^{n+N-1} is covered at least twice by $\tilde{\nu}$. In fact, for a fixed unit vector ν_0, the scalar product $\nu_0 \cdot x(p)$ as a continuous function on M^n has at least one maximum and one minimum, at which $\pm \nu_0 dx(p) = 0$. If, on the other hand, the maximum and minimum points are the same then M^n is contained in a hyperplane perpendicular to ν_0, and every point of M^n has ν_0 as a normal vector.

The set of critical points of $\tilde{\nu}$ is the set E of points in B_ν such that $G(p, \nu) = 0$. Hence

$$\int_{M^n} K^*(p)\, dV = \int_{B_\nu} |\, G(p, \nu)\,|\, dV d\sigma_{N-1}$$

is the volume of the image in S_0^{n+N-1} of the set of non-critical points of B_ν. By Sard's theorem and by the above remark that every point of S_0^{n+N-1} is covered at least twice by $\tilde{\nu}$, we have immediately

$$\int_{M^n} K^*(p)\, dV \geqq 2c_{n+N-1}.$$

4. Proof of Theorem 2. Suppose that $\int_{M^n} K^*(p)\, dV < 3c_{n+N-1}$; we wish to show that M^n is homeomorphic to a sphere of n dimensions. Our hypothesis implies that there exists a set of positive measure on S_0^{n+N-1} such that if ν_0 is a unit vector in this set, $\nu_0 \cdot x(p)$ has just two critical points. For if not; every point of S_0^{n+N-1}, except for a set of measure zero, would be covered at least three times by ν and hence as in Section 3 we would have

$$\int_{M^n} K^*(p)\, dV \geqq 3c_{n+N-1}.$$

Since, by Sard's theorem, the image of the set of critical points under $\tilde{\nu}$ is of measure zero, there is a unit vector ν_0 such that $\nu_0 \cdot x(p)$ has exactly two critical points on M^n and such that ν_0 is the image of non-critical points of B_ν under $\tilde{\nu}$. The latter means that $G(p, \nu_0) \neq 0$ at each critical point $p \in M$ of the function $\nu_0 \cdot x(p)$, which is equivalent to saying that $\nu_0 \cdot d^2x$ is a quadratic differential form of determinant not zero. In other words, the function $\nu_0 \cdot x(p)$ on M^n has exactly two non-degenerate critical points. Now a theorem of Reeb [5] (see also [4], p. 401) asserts that if a compact differentiable manifold M has a real-valued differentiable function on it with only two critical points which are non-degenerate, then M is homeomorphic to a sphere. It follows from this that M^n is homeomorphic to a sphere, and our theorem is proved.

5. Proof of Theorem 3. We begin with two lemmas:

LEMMA 1. *Under the hypothesis of Theorem* 3, M^n *is immersed in a linear subvariety of dimension* $n+1$ *of* E^{n+N}.

We may assume $N \geqq 2$, since otherwise there is nothing to prove. We will first show that M is contained in a hyperplane of E^{n+N}. In doing this, we show that the hypothesis of Theorem 3 and the assumption that M^n does not lie in a hyperplane of E^{n+N} lead to a contradiction.

Let $(p, \nu_0(p)) \in B_\nu$ be such that $G(p, \nu_0) \neq 0$; such a point exists by Theorem 1. Choose a local cross-section of M^n in B; i.e., the vectors $\tilde{e}_A(q)$, where q belongs to a neighborhood of p, and that $\tilde{e}_{n+N}(p) = \nu_0$. Then any other unit normal vector at p may be written $\nu = \sum_r \nu_r \tilde{e}_r(p)$ and by equation (21), Section 3, we have $G(p, \nu) = (-1)^n \det\left(\sum_r \nu_r \tilde{A}_{rij}(p)\right)$. Holding p fixed and restricting ourselves to normal vectors $\nu(\theta)$ such that

$$\nu_{n+N} = \cos\theta, \quad \nu_{n+N-1} = \sin\theta, \qquad \nu_r = 0, r \neq n+N-1, n+N,$$

we have $G(p, \nu) = f(\theta)$, where $f(\theta)$ is a polynomial in $\cos\theta$ and $\sin\theta$ and is hence an analytic function of θ. $f(\theta)$ is not identically zero, since $f(0) = G(p, \nu_0) \neq 0$.

Let H_θ be the tangent hyperplane at $x(p)$ perpendicular to $\nu = \nu(\theta)$. Since $x(M^n)$ does not belong to a hyperplane, there exist tangent hyperplanes H_{θ_1}, H_{θ_2}, $\theta_1 < \theta_2$, and points $q_1, q_2 \in M$, such that $x(q_1) \in H_{\theta_1}$, $x(q_2) \in H_{\theta_2}$, $x(q_1) \notin H_{\theta_2}$, $x(q_2) \notin H_{\theta_1}$. Since $f(\theta)$ does not vanish identically, there is a tangent hyperplane H_{θ_3}, such that $f(\theta_3) \neq 0$ and that $x(q_1)$ and $x(q_2)$ lie on different sides of the tangent hyperplane H_{θ_3}. The condition $f(\theta_3) \neq 0$ implies that the mapping $\tilde{\nu}$ is one-one in a neighborhood W of $(p, \nu(\theta_3))$ of B_ν. We can choose W so small that for $(q', \nu') \in W$, $x(q_1)$ and $x(q_2)$ lie on different sides of the tangent hyperplane perpendicular to ν' at $x(q')$. The function $\nu' \cdot x$ in M^n has at least three point at which $\nu' \cdot dx = 0$; namely, the maximum, the minimum, and the point q'. The last point q' is distinct from the maximum and the minimum, since, by our construction, there are points of $x(M^n)$ on different sides of the tangent hyperplane at q' perpendicular to ν'. It follows that a neighborhood of ${S_0}^{n+N-1}$ is covered by the image of $\tilde{\nu}$ at least three times. As every point of ${S_0}^{n+N-1}$ is covered at least twice, we conclude that the total curvature of $x(M^n)$ is strictly greater than $2c_{n+N-1}$. But this is a contradiction.

It follows from this contradiction that $x(M^n)$ belongs to a hyperplane E^{n+N-1} of E^{n+N}. We wish to show that its total curvature in E^{n+N-1} is equal

to $2c_{n+N-2}$. We denote by v the unit vector perpendicular to E^{n+N-1} and by S_0^{n+N-2} the unit sphere in E^{n+N-1}. The sphere S_0^{n+N-2} can be imagined as the equator of S^{n+N-1} with v as the north pole. Let B_ν' be the bundle of unit normal vectors of $x(M^n)$ in E^{n+N-1}. Then $B_\nu' \subset B_\nu$ and $\nu(B_\nu') \subset S_0^{n+N-2}$. Denote by $\bar{\nu}'$ the restriction of $\bar{\nu}$ to B_ν'. It suffices to prove that, with the exception of a set of measure zero on S_0^{n+N-2}, the points of S_0^{n+N-2} are covered by $\bar{\nu}'$ exactly twice.

Suppose the contrary be true. There is thus a set A of positive measure on S_0^{n+N-2}, whose points are covered by $\bar{\nu}'$ more than twice. To any $\mu \in A$ there are distinct points $p_1, \cdots, p_k \in M^n$, $k \geqq 3$, for which $x(p_1), \cdots, x(p_k)$ have normal vectors parallel to μ. All the unit vectors in the great circle spanned by μ and v are then normal to $x(M^n)$ at $x(p_1), \cdots, x(p_k)$. It follows that all the points of S_0^{n+N-1} belonging to great circles spanned by v and the points of A are covered by $\bar{\nu}$ more than twice. Since A has positive measure on S_0^{n+N-2}, this set has positive measure on S_0^{n+N-1}, which is a contradiction.

Therefore $x(M^n)$ belongs to E^{n+N-1} and has, as a submanifold of E^{n+N-1}, a total curvature equal to $2c_{n+N-2}$. By induction on N we see that $x(M^n)$ must belong to a linear subvariety E^{n+1} of dimension $n+1$ and has in E^{n+1} the total curvature $2c_n$.

LEMMA 2. *Let $x: M^n \to E^{n+1}$ be an immersion of a compact oriented manifold and let $\nu: M^n \to S_0^n$ be the normal map of Gauss. Let $J(p)$ be the Jacobian matrix of ν at p, and let $U_m = \{p \in M^n \mid \text{rank } J(p) = n - m\}$. Then, if U_m contains an open set V, its image under x is generated by m-dimensional planes. Every boundary point of U_m, which is at the same time a limit point of an m-dimensional generating plane, belongs to U_m.*

The fact that the image under x of V is generated by m-dimensional planes is a classical result; we include a proof for completeness. At any interior point p of U_m the assumption on the rank of J implies that we may choose coordinates on M^n in the neighborhood of p, say, $t_1, \cdots, t_n$, such that if ν is the unit normal vector at p, then $\partial\nu/\partial t_\alpha = 0$, and $\partial\nu/\partial t_a$ are linearly independent. Here, and for the remainder of this section, we make the following convention concerning indices:

$$1 \leqq \alpha, \beta, \gamma \leqq m, \qquad m+1 \leqq a, b, c \leqq n, \qquad 1 \leqq i, j, k \leqq n.$$

We have $\nu \cdot \partial x/\partial t_i = 0$. It follows that

$$\nu \cdot \partial^2 x/\partial t_i \partial t_\alpha = 0, \quad \partial\nu/\partial t_\alpha \cdot \partial x/\partial t_i + \nu(\partial^2 x/\partial t_i \partial t_\alpha) = 0, \quad \partial\nu/\partial t_a \cdot \partial x/\partial t_\alpha = 0.$$

Hence $\partial x/\partial t_\alpha$ are vectors orthogonal to the $n-m+1$ linearly independent

vectors ν, $\partial/\partial t_a$. The surfaces $t_a = t_a^0 = \text{const.}$ are therefore m-dimensional planes in E^{n+1}. Since $\partial\nu/\partial t_\alpha = 0$, the tangent hyperplane remains constant along an m-dimensional generating plane.

Consider now the sub-bundle B' of B consisting of all those frames $e_1, \cdots, e_{n+1}$ such that e_α are in the m-dimensional generating planes. Then, as in Section 1, we have

$$dx = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \omega_{i,n+1} e_{n+1}, \quad de_{n+1} = \sum_j \omega_{n+1,j} e_j,$$

where

$$\omega_{i,n+1} = \sum_j A_{ij}\omega_j, \qquad A_{ij} = A_{ji}.$$

The above assumption on the bundle of frames B' is equivalent to assuming

$$\omega_{\alpha,n+1} = -\omega_{n+1,\alpha} = 0; \tag{24}$$

i.e., that the matrix (A_{ij}) takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{ab} \end{pmatrix}, \qquad \det(A_{ab}) = D \neq 0,$$

which is an $(n \times n)$-matrix whose elements are zero, except possibly those of the $(n-m)\times(n-m)$ block in the lower right-hand corner. Our proof depends on studying the behavior of D along the m-dimensional generating plane.

From (24) we have

$$0 = d\omega_{\alpha,n+1} = \sum_k \omega_{\alpha k} \wedge \omega_{k,n+1} = \sum_\beta \omega_{\alpha\beta} \wedge \omega_{\beta,n+1} + \sum_a \omega_{\alpha a} \wedge \omega_{a,n+1}.$$

Our assumption implies

$$\omega_{\beta,n+1} = 0, \qquad \omega_{a,n+1} = \sum_b A_{ab}\omega_b,$$

so that $\sum_{a,b} A_{ab}\omega_{\alpha a} \wedge \omega_b = 0$, or $\sum_a A_{ab}\omega_{\alpha a} \wedge \prod_c \omega_c = 0$. Since $\det(A_{ab}) \neq 0$, we get $\omega_{\alpha a} \wedge \prod_c \omega_c = 0$. Hence we can put

$$\omega_{\alpha a} = \sum_b h_{\alpha a b}\omega_b. \tag{25}$$

Now we have $\prod_a \omega_{a,n+1} = D \prod_c \omega_c$. Exterior differentiation of this equation gives

$$\sum_a (-1)^{a-m-1} \omega_{m+1,n+1} \wedge \cdots \wedge d\omega_{a,n+1} \wedge \cdots \wedge \omega_{n,n+1}$$
$$= dD \wedge \prod_c \omega_c + D\Big(\sum_a (-1)^{a-m-1} \omega_{m+1} \wedge \cdots \wedge d\omega_a \wedge \cdots \wedge \omega_n\Big).$$

But

$$d\omega_{a,n+1} = \sum_k \omega_{ak} \wedge \omega_{k,n+1} = \sum_b \omega_{ab} \wedge \omega_{b,n+1},$$

$$d\omega_a = \sum_k \omega_k \wedge \omega_{ka} = \sum_{a,b} h_{aab}\omega_a \wedge \omega_b + \sum_b \omega_b \wedge \omega_{ba}.$$

Hence $0 = dD \wedge \prod_c \omega_c + D(\sum_{a,\alpha} h_{aa\alpha}\omega_\alpha \wedge \prod_c \omega_c)$, or

$$dD + D(\sum_{a,\alpha} h_{aa\alpha}\omega_\alpha) \equiv 0, \mod \omega_c. \tag{26}$$

To complete the proof let $p \in M^n$ be a boundary point of U_m, such that $x(p)$ is a limit point of a generating m-dimensional plane L. We choose a neighborhood V of p, in which x is one-one, and we suppose that $x^{-1}(L) \subset V$. Let $\tilde{e}_1(q), \cdots, \tilde{e}_{n+1}(q)$, $q \in V$, be a local cross-section of V into B, such that, for $q \in x^{-1}(L)$, $\tilde{e}_1(q), \cdots, \tilde{e}_m(q)$ span L. Such a cross-section clearly exists. If $\tilde{\omega}_i$, $\tilde{\omega}_j$ are the restrictions of ω_i, ω_{ij} respectively to this cross-section, then $\tilde{\omega}_i$ are linearly independent and we will have

$$\tilde{\omega}_{a\alpha} = \sum_b h_{a\alpha b}\tilde{\omega}_b.$$

The coefficients $\tilde{h}_{a\alpha b}(q)$ are equal to the functions $h_{a\alpha b}(q)$ introduced above, for $q \in x^{-1}(L)$. Let γ be a curve in $x^{-1}(L)$ abutting at p. We have, along γ,

$$dD + D(\sum_{a,\alpha} h_{aa\alpha}\omega_\alpha) = 0.$$

It follows by integration of this differential equation that

$$D(q) = D_0 \exp(-\int \sum h_{aa\alpha}\omega_\alpha),$$

for $q \in \gamma - p_0$, where $D_0 \neq 0$ is the value of D at a fixed point of γ. Since $D(q)$ is a continuous function and since $h_{aa\alpha}$ is bounded, we conclude that $D(p) \neq 0$. This completes the proof of Lemma 2.

We now complete the proof of Theorem 3 as follows:

Let H be the space of hyperplanes of E^{n+1} (with the obvious topology). A tangent hyperplane of $x(M^n)$ is said to be of rank m, if it is tangent to $x(M^n)$at a point of $x(U_m)$ and at no point of $x(U_l)$, $l < m$. By the argument used in the proof of Theorem 2, a tangent hyperplane of rank zero does not separate the set $x(M^n)$. For otherwise there would be a neighborhood of S_0^n, whose points are covered at least three times by ν, which would be contrary to the assumption that M^n has the minimum total curvature $2c_n$.

We will show that in every neighborhood (in the space H) of a tangent hyperplane π of $x(M^n)$ there is a tangent hyperplane of rank zero. In fact, let W be such a neighborhood. Suppose $x(p)$, $p \in U_m$, be a point of contact of π. Either there is a neighborhood of p in M which belongs completely to U_m or there are points of U_l, $l < m$, in every neighborhood of p. In both cases we can find a point p_1 such that the tangent hyperplane π_1 at $x(p_1)$ belongs to W and such that p_1 has a neighborhood in M^n which belongs completely to U_l, $l \leqq m$. The image under x of this neighborhood of p_1 is

generated by l-dimensional planes and the tangent hyperplane to $x(M^n)$ along the generating l-dimensional plane through $x(p_1)$ is π_1. If $x(p_2)$, $p_2 \in M^n$, is a boundary point of this l-plane, p_2 belongs to U_l by Lemma 2 and is not an interior point of U_l. Hence there exists in every neighborhood about p_2 an open set whose points are in U_k, $k < l$, and which contains a point $p_3 \in U_k$ such that the tangent hyperplane at $x(p_3)$ is in W. Continuing this process, we see that W contains a tangent hyperplane of rank zero of $x(M^n)$.

This means that every neighborhood of π in H contains a tangent hyperplane such that $x(M^n)$ lies on one side. It follows that the same is true for π itself. If ν_0 is any unit vector in E_0^{n+1}, $\nu_0 \cdot x(p)$ has a maximum and a minimum on M^n, which must be distinct, since M^n cannot be immersed in an m-dimensional hyperplane. Hence the intersection of all the half-spaces of E^{n+1} bounded by a tangent hyperplane of $x(M^n)$ and containing points of $x(M^n)$ is a closed convex set with a non-empty interior and with $x(M^n)$ on the boundary.

Since the induced homomorphism x_* of tangent spaces is one-one, x is a local homeomorphism of M into the boundary of the convex set. It follows that $x(M^n)$ is both open and closed on the boundary; thus x maps M^n onto the boundary. But the boundary of the convex set is homeomorphic to a sphere S^n and by the above M^n is a covering space of S^n under the map x. Hence, if $n \geqq 2$, x is a homeomorphism. The same is true for $n = 1$, on account of the fact that the total curvature is $2c_1$.

Conversely, let $x(M^n)$ be a convex hypersurface. It is then locally convex. By reversing the orientation of M^n if necessary, we can suppose that $G(p) \geqq 0$. Then $K^*(p) = 2G(p)$, because there are two unit normal vectors at every point. The degree of ν is

$$1/c_n \int_{M^n} G(p)\, dV = 1.$$

Hence the total curvature of $x(M^n)$ is $2c_n$.

6. A further theorem.

THEOREM 4. *Let $x: M^n \to E^{n+1}$ be an immersion of a closed orientable manifold and $\nu: M^n \to S_0^n$ the normal map. Then the following are equivalent:*

1. $\deg \nu = \pm 1$ *and the Gaussian curvature is of constant sign;*
2. *The total curvature is $2c_n$;*
3. *M^n is imbedded as a convex hypersurface.*

It suffices to prove the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1). Since 2) $\Rightarrow$ 3) $\Rightarrow$ 1) are contained in Theorem 3, we only have to prove that 1) implies 2).

For this purpose it is sufficient to show that no set V of positive measure on S_0^n is covered more than once by M^n under ν. Suppose the contrary. By reversing the orientation of M^n if necessary, we can suppose that $\deg \nu = +1$ and that the Gaussian curvature is non-negative. By Sard's theorem, there exists a point $y \in V$ such that for any point in $\nu^{-1}(y)$ the curvature is strictly positive. There can only be a finite number of points in $\nu^{-1}(y)$. For otherwise $\nu^{-1}(y)$ will have a limit point p at which ν is not locally one-one, while, on the other hand, the Jacobian of ν at $p \in \nu^{-1}(y)$, being a non-zero multiple of the Gaussian curvature at p, is different from zero. By our assumption on V the number of points in $\nu^{-1}(y)$ is $\geqq 2$. At each point of $\nu^{-1}(y)$ the Jacobian of ν is strictly positive. It follows that $\deg \nu \geqq 2$, which contradicts our assumption. Hence the theorem is proved.

Remark. We would like to conjecture that for $n \geqq 2$ the condition $\deg \nu = 1$ in 1) can be omitted. In other words, it seems likely that a closed orientable hypersurface (of dimension $\geqq 2$) of non-negative Gauss-Kronecker curvature is convex. If the curvature is strictly positive, this follows from Hadamard's principle. On the other hand, it is well-known that this condition is essential for $n = 1$; there are non-convex immersed curves in the plane with non-negative curvature.

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SUR LES REVÊTEMENTS NON RAMIFIÉS DES VARIÉTÉS ALGÉBRIQUES.*

par SERGE LANG et JEAN-PIERRE SERRE.

Introduction. Soit V une variété algébrique normale définie sur le corps $\boldsymbol{C}$ des nombres complexes. Munissons V de la topologie "usuelle," déduite de celle de $\boldsymbol{C}$. Tout revêtement algébrique non ramifié de V peut être considéré comme un revêtement topologique, et correspond donc à un sous-groupe d'indice fini du groupe fondamental $\pi_1(V)$ de V. Des propriétés connues du groupe fondamental résulte alors immédiatement:

a) Si V et W sont deux variétés, tout revêtement de $V \times W$ est quotient d'un revêtement de la forme $V' \times W'$, où V' et W' sont des revêtements de V et de W.

(C'est une conséquence de l'égalité $\pi_1(V \times W) = \pi_1(V) \times \pi_1(W)$).

b) Le nombre des revêtements de V de degré donné est fini.

(C'est une conséquence du fait que $\pi_1(V)$ peut être engendré par un nombre fini d'éléments).

Dans ce qui suit, nous nous proposons de donner des démonstrations *algébriques* de a) et de b), valables en toute caractéristique (cf. th. 1 et th. 4); nous devons toutefois faire l'hypothèse supplémentaire que V est une variété *complète*. Des contre-exemples simples (dus essentiellement à la présence de "ramification supérieure") montrent d'ailleurs que a) et b) ne sont pas vrais sans restriction en caractéristique $p > 0$.

Comme application de a), nous démontrons que tout revêtement non ramifié d'une variété abélienne est une isogénie (th. 2), généralisant ainsi un résultat bien connu pour les courbes elliptiques.

1. Conventions et notations. Nous adopterons la définition des *revêtements* donnée dans [4] § 1, à cela près que nous ne nous occuperons que de revêtements *géométriques*. Par définition, un revêtement est donc une application rationnelle partout définie

$$f: U \to V,$$

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vérifiant les conditions suivantes:

α) U et V sont des variétés *normales* (absolument irréductibles) de même dimension.

β) Pour tout $v \in V$, l'ensemble $f^{-1}(v)$ est *fini*.

γ) La variété U est *complète* au-dessus de tout point de V.

δ) Si K et L désignent les corps des fonctions rationnelles sur V et sur U, l'extension L/K est *séparable*.

Si U, V et f sont définis sur un corps k, on dira plus brièvement que le revêtement est défini sur k.

Les propriétés ci-dessus entraînent que U est la normalisée de V dans l'extension L/K (pour tout ce qui concerne l'operation de normalisation, voir par exemple [6], Chap. II, § 5); inversement, si l'on se donne une variété normale V ayant K pour corps de fonctions, et si l'on se donne une extension finie séparable L de K, la normalisée U de V dans L/K est un revêtement de V. Ainsi, V étant donnée, il y a correspondance bijective entre revêtements de V et extensions finies séparables de K. Le *degré* de l'extension L/K sera appelé le degré du revêtement $f: U \to V$, et sera noté $[U: V]$. De même, un revêtement sera dit *galoisien*, de groupe de Galois G, si l'extension L/K correspondante est galoisienne et de groupe de Galois G; on a alors $V = U/G$, avec les notations de [7], no. 13.

Soit $f: U \to V$ un revêtement de degré n, défini sur un corps algébriquement clos k. Si v est un point de V, l'ensemble $f^{-1}(v)$ a au plus n éléments; s'il en a exactement n, on dira que U est *non ramifié en* v. L'ensemble des points de ramification est un sous-ensemble k-fermé de V; s'il est vide, on dira que U est *non ramifié*. En fait, pour qu'une application rationnelle partout définie $f: U \to V$ soit un revêtement non ramifié, il suffit, d'après Krull ([2], voir aussi [4], p. 292), que V soit normale, que les conditions β) et γ) soient remplies, et que pour tout point $v \in L$ l'ensemble $f^{-1}(v)$ ait exactement n points (n désignant comme précédemment le degré de l'extension L/K); la variété U est alors automatiquement normale, et l'extension L/K séparable.

Revenons au cas d'un revêtement quelconque $f: U \to V$, et soit V' une sous-variété de V. Soit $f^{-1}(V')$ l'image réciproque de V' dans U, et soient U_i' les composantes irréductibles de $f^{-1}(V')$. En appliquant les théorèmes du type Cohen-Seidenberg, on voit tout de suite que les U_i' ont même dimension que V'; de plus, si l'on note $[U_i': V']$ le degré de la projection $U_i' \to V'$, on a:

$$\Sigma [U_i': V'] \leqq [U: V],$$ cf. [2] ainsi que [4], *loc. cit.*.

Lorsque tous les degrés $[U_i' : V']$ sont égaux à 1, on dira que le revêtement U *se décompose complètement* sur V'.

En général, les U_i' ne sont pas des variétés normales, et les extensions de corps correspondant aux projections $U_i' \to V'$ ne sont pas séparables. On a toutefois:

LEMME 1. *Supposons que le revêtement $f: U \to V$ soit non ramifié, et soit V' une sous-variété normale de V. Alors les composantes U_i' de $f^{-1}(V')$ sont des variétés normales et disjointes. Chacune d'elles est un revêtement non ramifié de V', et l'on a l'égalité:*

$$\Sigma[U_i' : V'] = [U : V].$$

Posons $n = [U : V]$ et $n_i = [U_i' : V']$; d'après ce qui a été dit plus haut, on a en tout cas $\Sigma n_i \leqq n$. D'autre part, soit $v \in V'$, et soit m_i le norbre des points de U_i' se projetant en v; du fait que V' est normale, on a $m_i \leqq n_i$. Mais d'autre part, puisque $f: U \to V$ est non ramifié, il a exactement n points de U qui se projettent en v. On a donc $\Sigma m_i \geqq n$, d'où $m_i = n_i$ et $\Sigma n_i = n$. D'après le résultat de Krull déjà cité, l'égalité $m_i = n_i$ entraîne que U_i' est un revêtement non ramifié de V', et en particulier que U_i' est une variété normale; enfin, si U_i' et U_j' avait en commun un point u, de projection v, l'ensemble des points de U se projetant en v aurait au plus $\Sigma m_i - 1 = n - 1$ éléments, ce qui est impossible; les variétés U_i' sont donc bien disjointes, ce qui achève la démonstration.

(Notons que, si $f: U \to V$ est galoisien, il en est de même de chacun des revêtements $U_i' \to V'$; leurs groupes de Galois sont des sous-groupes de celui de U—les classiques "groupes de décomposition").

Plus généralement, soit $\alpha: V' \to V$ une application partout définie d'une variété V' dans V; supposons encore que V' soit normale. Alors $V' \times U$ forme de façon naturelle un revêtement de $V' \times V$; si l'on plonge V' dans $V' \times V$ au moyen du graphe de α, on peut encore définir $f^{-1}(V') \subset V' \times U$, et l'on se retrouve dans la situation envisagée ci-dessus; dans la suite, nous noterons $\alpha^*(U)$ l'ensemble algébrique $f^{-1}(V')$; par définition, c'est l'ensemble des couples $(v', u) \in V' \times U$ tels que $\alpha(v') = f(u)$. Lorsque U est non ramifié, le lemme 1 montre que les composantes irréductibles de $\alpha^*(U)$ sont des revêtements non ramifiés de V' (les "pull-back" de [4], § 4).

2. Revêtements d'un produit de variétés. Soient V et W deux variétés normales. Dans ce no., nous aurons à considérer des revêtements du produit $V \times W$. Si $f: U \to V \times W$ est un tel revêtement, et si w est un

point de W, nous noterons U_w l'image réciproque de $V \times \{w\}$ dans U; si l'on note α_w l'injection de V dans $V \times W$ définie par la formule $\alpha_w(v) = (v, w)$, l'ensemble algébrique U_w n'est autre que $\alpha_w^*(U)$. Lorsque U est non ramifié, les composantes irréductibles de U_w sont des revêtements non ramifiés de V, d'après le lemme 1.

LEMME 2. *Soit k un corps de définition d'un revêtement $f: U \to V \times W$, et soit w un point générique de W sur k. Supposons que U soit complètement décomposé sur $V \times \{w\}$. Alors U est isomorphe à un revêtement $V \times W'$, où W' est un revêtement de W.*

Soit v un point générique de V, indépendant de w, et soit $u \in U$ tel que $f(u) = (v, w)$; le point u est alors un point générique de U sur k. On voit tout de suite que le lieu de u sur le corps $k(w)$ n'est autre que U_w, qui est donc irréductible sur $k(w)$. Soit L la fermeture algébrique de $k(w)$ dans $k(u)$; par hypothèse, $k(u)/k(v, w)$ est séparable, et il en est évidemment de même de $k(v, w)/k(w)$; donc $k(u)/k(w)$ est séparable, et l'extension $k(u)/L$ est régulière. Il s'ensuit que le lieu de u sur L est l'une des composantes irréductibles de U_w; puisque on a supposé U complètement décomposé sur $V \times \{w\}$, on a donc $k(u) = L(v)$. Si l'on désigne alors par W' la normalisée de W dans l'extension $L/k(w)$, la variété $V \times W'$ est la normalisée de $V \times W$ dans $k(u)/k(v, w)$, donc est isomorphe à U, cqfd.

Remarque. Le lemme 2 ne s'étend pas aux "revêtements inséparables" (c'est-à-dire ceux où l'on supprime l'hypothèse δ) du no. 1). Pour le voir, il suffit de poser $k(w) = k(x, y)$ et $k(u) = k(v, x, y, \theta)$, avec $\theta^p = x + v^p y$, k désignant un corps algébriquement clos de caractéristique p.

LEMME 3. *Soit $f: U \to V \times W$ un revêtement non ramifié, V étant un variété complète. Supposons qu'il existe $a \in W$ tel que U se décompose complètement sur $V \times \{a\}$. Alors U est isomorphe à un revêtement $V \times W'$, où W' est un revêtement non ramifié de W.*

Supposons d'abord que V soit une variété *projective*. Soit k un corps de définition du revêtement, et soit w un point générique de W par rapport à k. D'après le théorème de dégénérescence de Zariski ([8]), le nombre de composantes connexes de U_w est au moins égal à celui de sa spécialisation U_a; mais, d'après le lemme 1, ce dernier nombre est égal à celui des composantes irréductibles de U_a, lequel est égal à $[U : V]$ par hypothèse. Il s'ensuit que le nombre de composantes connexes de U_w est aussi égal à $[U : V]$, ce qui signifie que U se décompose complètement sur $V \times \{w\}$;

d'après le lemme 2, il s'ensuit que $U = V \times W'$, et on voit immédiatement que W' est non ramifié.

L'hypothèse que V est projective n'a été appliquée que pour pouvoir utiliser le théorème de dégénérescence de Zariski. Le cas général se ramène au cas projectif: d'après un résultat de Chow (cf. [1]), il existe une variété projective normale V_1 et une application birationnelle partout définie

$$\phi: V_1 \to V.$$

L'application ϕ définit une application $\psi: V_1 \times W \to V \times W$; posons $U_1 = \psi^*(U)$; en appliquant la première partie de la démonstration à U_1, on voit que $U_1 = V_1 \times W'$. Le corps $k(U_1)$ des fonction rationnelles sur U_1 est donc un composé $k(V_1) \cdot k(W') = k(V) \cdot k(W')$; il s'ensuit que $k(U) = k(V) \cdot k(W')$, d'où $U = V \times W'$, cqfd.

Remarque. Le lemme 3 est inexact lorsque l'on ne suppose plus que V est complète: on le voit en prenant pour V et W des droites affines de points génériques v et w, et en définissant U par l'équation $u^p - u = vw$ (si l'on fait $v = 0$ ou $w = 0$, l'équation se décompose, ce qui serait incompatible avec le lemme 3).

THÉORÈME 1. *Soit $f: U \to V \times W$ un revêtement non ramifié, V étant une variété complète. Il existe alors deux revêtements non ramifiés V' de V et W' de W tels que U soit un quotient du revêtement produit $V' \times W'$.*

Nous pouvons évidemment supposer que le revêtement U est galoisien; soit G son groupe de Galois. Choisissons alors un point $a \in W$, et soit $U_a = f^{-1}(V \times \{a\})$; d'après le lemme 1, les composantes irréductibles de U_a sont des revêtements non ramifiés de V; comme on a supposé U galoisien, il en sera de même pour ces composantes (leurs groupes de Galois étant des sous-groupes de G, les "groupes de décomposition"). Choisissons l'une de ces composantes, soit V', et soit $G' \subset G$ son groupe de Galois. La projection $V' \to V$ définit une projection

$$h: V' \times W \to V \times W,$$

qui fait de $V' \times W$ un revêtement de $V \times W$ de groupe de Galois G'. Posons $U = h^*(U)$, sous-ensemble algébrique de $V' \times W \times U$; comme précédemment pour U_a, les composantes irréductibles de U_1 sont des revêtements non ramifiés de $V' \times W$, galoisiens et de groupes de Galois des sous-groupes de G. Soit $b \in V'$ et soit U' la composante de U_1 qui contient le point $(b, a, b) \in U_1$; comme U' est un revêtement non ramifié de $V' \times W$, c'est

aussi un revêtement non ramifié de $V \times W$; de plus, puisque la projection de U' sur $V \times W$ se factorise en $U' \to U \to V \times W$, le revêtement U est un quotient de U', et il nous suffira donc de prouver que U' est de la forme $V' \times W'$. Mais U' se décompose complètement sur $V' \times \{a\}$: en effet, U_1 contient la sous-variété de $V' \times W \times U$ formée des points (v', a, v') où v' parcourt V'; comme cette sous-variété a le point (b, a, b) en commun avec U', elle est contenue dans U', et constitue une composante de U_a' se projetant sur $V' \times \{a\}$ avec degré 1; du fait que U' est galoisien, ceci suffit à assurer que U' se décompose complètement sur $V' \times \{a\}$. En appliquant alors le lemme 3, on en déduit bien que U' est isomorphe à un revêtement de la forme $V' \times W'$, ce qui achève la démonstration.

COROLLAIRE 1. *Soit G un groupe abélien fini. Les notations étant celles de* [7], no. 14, *on a* $\pi^1(V \times W, G) = \pi^1(V, G) \times \pi^1(W, G)$.

C'est immédiat.

COROLLAIRE 2. *Les hypothèses étant celles du théorème* 1, *soient a et b deux points de W. Il existe alors un isomorphisme de l'ensemble algébrique U_a sur l'ensemble algébrique U_b compatible avec les projections $U_a \to V$ et $U_b \to V$.*

(On a un énoncé analogue en permutant les rôles de V et de W).

En effet, d'après le théorème 1, le revêtement U est quotient d'un revêtement $g: V' \times W' \to V \times W$; on peut supposer V' et W' galoisiens de groupes de Galois G_1 et G_2 respectivement; alors U est isomorphe à $(V' \times W')/H$, où H est un sous-groupe de $G_1 \times G_2$. Posons:

$$U_a' = g^{-1}(V \times \{a\}) \quad \text{et} \quad U_b' = g^{-1}(V \times \{b\}).$$

Il est clair que U_a' et U_b' sont tous deux isomorphes à $V' \times G_2$, ces isomorphismes commutant avec les opérations de $G_1 \times G_2$; comme U_a est isomorphe à U_a'/H, il est isomorphe à $(V' \times G_2)/H$, et de même pour U_b, ce que démontre le corollaire.

Lorsque U est galoisien, de groupe de Galois G, on a $G = (G_1 \times G_2)/H$; de plus, on peut supposer (quitte à changer V' et W') que $G_1 \cap H = \{e\}$ et $G_2 \cap H = \{e\}$. Les groupes G_1 et G_2 s'identifient ainsi à des sous-groupes de G, invariants, commutant entre eux, et engendrant G. De plus, le raisonnement fait ci-dessus montre que, pour tout $a \in W$, l'ensemble algébrique U_a se décompose en composantes irréductibles qui sont isomorphes à V', le groupe G_1 s'identifiant au groupe de décomposition correspondant; on a un résultat analogue pour U_v si $v \in V$. En définitive, on a obtenu:

COROLLAIRE 3. *Outre les hypothèses du théorème* 1, *supposons que* U *soit galoisien de groupe de Galois* G. *Soit* $(v, w) \in V \times W$, *et soit* V' (*resp.* W') *une composante de* U_w (*resp.* U_v); *soit* G_1 (*resp.* G_2) *son groupe de Galois. Les groupes* G_1 *et* G_2 *sont alors des sous-groupes invariants de* G, *qui commutent entre eux et engendrent* G; *si* H *désigne le noyau de l'homomorphisme* $G_1 \times G_2 \to G$, *le revêtement* U *est isomorphe à* $(V' \times W')/H$.

3. Revêtements non ramifiés des variétés abéliennes. Nous nous proposons de démontrer le résultat suivant:

THÉORÈME 2. *Toute revêtement non ramifié d'une variété abélienne est une isogénie* (*séparable*).

(Cela montre en particulier qu'un tel revêtement est *abélien*).

Soit donc V une variété abélienne, et soit $f: U \to V$ un revêtement non ramifié de V. On peut évidemment supposer que U est galoisien; soit G son groupe de Galois. Il nous faut montrer que l'on peut définir sur U une structure de variété abélienne (auquel cas f sera automatiquement une isogénie).

Soit $g: V \times V \to V$ l'application qui définit la loi de composition de V, i.e. $g(v, v') = v + v'$. Soit $U' = g^*(U)$, sous-ensemble algébrique de $V \times V \times U$ formé des triples (v, v', u) tels que $f(u) = v + v'$. Le groupe G permute les composantes irréductibles de U'; mais si l'on pose

$$U_1' = U' \cap (\{0\} \times V \times U),$$

l'ensemble U_1' s'identifie à U, donc est irréductible; *a fortiori*, il en est de même de U'. Nous noterons $f': U' \to V \times V$ la projection de U' sur les deux premiers facteurs du produit $V \times V \times U$. On a évidemment $U_1' = f'^{-1}(\{0\} \times V)$, et l'on vient de voir que U_1' s'indentifie à U; de même $U_2' = f'^{-1}(V \times \{0\})$ s'identifie à U. Appliquant alors le corollaire 3 au théorème 1, on voit que. avec les notations de ce corollaire, on a $G_1 = G_2 = G$, ce qui montre d'abord que G est *abélien*, puis que U' s'identifie à $(U \times U)/H$, où H désigne le sous-groupe de $G \times G$ formé par les éléments (g, g^{-1}), $g \in G$. Comme on a une application canonique $h: U' \to U$, on en déduit par composition une application $g': U \times U \to U' \to U$ qui "relève" l'application $g: V \times V \to V$. Si l'on a choisi dans U un point e tel que $f(e) = 0$, on peut en outre s'arranger pour que $g'(e, e) = e$. Il ne nous reste plus qu'à démontrer que l'application g' vérifie les axiomes d'une loi de groupe, ce que ne présente pas de difficultés:

a) Le point $e \in U$ est élément neutre pour la loi de composition g'. En effet, soit $\sigma: U \to U$ l'application définie par la formule:

$$\sigma(u) = g'(e, u).$$

On a $f \circ \sigma = f$, ce qui montre que σ est un automorphisme du revêtement U, i.e. un élément du groupe de Galois G. Mais on a $\sigma(e) = e$, et comme G opère sans points fixes, cela montre que σ est l'identité; un raisonnement analogue montre que $g'(u, e) = u$ pour tout $u \in U$.

b) La loi de composition g' est associative.

Soient k_1 et k_2 les applications de $U \times U \times U$ dans U définies par les formules:

$$k_1(v, v', v'') = g'(v, g'(v', v'')) \quad \text{et} \quad k_2(v, v', v'') = g'(g'(v, v'), v'').$$

Du fait que g' relève g, les projections de $k_1(v, v', v'')$ et de $k_2(v, v', v'')$ dans U sont les mêmes. On a donc $k_2(v, v', v'') = \sigma \cdot k_1(v, v', v'')$, avec $\sigma \in G$. Appliquons ceci à trois points génériques indépendants v, v', v''; en spécialisant v, v' et v'' en e, on voit que $\sigma = 1$, d'où $k_1 = k_2$ et g' est bien une loi de composition associative.

c) Il existe un inverse $u \to u^{-1}$ qui est un automorphisme de U.

Désignons par f'' l'application $f \times f: U \times U \to V \times V$; soit d'autre part $\phi: V \times V \to V \times V$ l'application définie par $(v, v') \to (v, v + v')$, et soit de même $\psi: U \times U \to U \times U$ l'application définie par

$$\psi(u, u') = (u, g'(u, u')).$$

On a évidemment la relation de commutation $f'' \circ \psi = \phi \circ f''$. En prenant les degrés des deux membres, et en tenant compte du fait que ϕ est birationnelle, on voit qu'il en est de même de ψ. De plus, ψ est partout définie puisque ϕ l'est. L'existence d'un inverse, et le fait que $u \to u^{-1}$ est un automorphisme résultent aussitôt de là, ce qui achève la démonstration.

Remarques.

1) La démonstration précédente est calquée sur la démonstration classique prouvant que tout revêtement d'un groupe topologique est un groupe topologique (démonstration qui repose elle-même sur la formule $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$).

2) Le théorème 2 *ne s'étend pas* aux groupes algébriques non complets. On trouvera des contre-exemples dans [3], relatifs au cas où V est un groupe

non commutatif, défini sur un corps fini. En fait, ou peut même construire des contre-exemples où V est le groupe additif G_a: si U' est une courbe elliptique d'invariant de Hasse nul, une différentielle de première espèce de U' peut s'écrire $\omega = df$, où f est une fonction rationnelle; soit U l'ouvert de U' obtenu en retirant l'ensemble des pôles de f; il est clair que $f: U \to G_a$ est un revêtement non ramifié qui n'est pas une isogénie.

4. Finitude du nombre des classes de revêtements non ramifiés d'une variété complète. Nous démontrerons d'abord le résultat suivant:

THÉORÈME 3. *Soit V une variété normale, complète, définie sur un corps algébriquement clos k. Alors tout revêtement non ramifié de V est isomorphe à un revêtement défini sur k.*

(Autrement dit, il n'y a pas de "systèmes algébriques" de revêtements non ramifiés).

Soit $f_1: U_1 \to V$ un revêtement non ramifié de V, défini sur une extension L/k que l'on peut supposer de type fini. Soient v un point générique de V sur L et u un point de U_1 tel que $f_1(u) = v$. Soit d'autre part W' une variété normale, définie sur k, telle que, si w est un point générique de W' sur k, on ait $L = k(w)$. On a $L(v) = k(v, w)$, corps des fonctions rationnelles sur $V \times W'$, et le corps $L(u)$ est une extension finie de $k(v, w)$. Soit U' la normalisée de $V \times W'$ dans l'extension $L(u)/k(v, w)$, et soit f la projection canonique de U' sur $V \times W'$. It est clair que $U_w' = f^{-1}(V \times \{w\})$ est une variété $k(w)$-normale, et en correspondance birationnelle, sur $k(w)$, avec le revêtement donné U_1; comme U_1 est normale, ceci montre que U_w' et U_1 sont isomorphes. Soit $Z \subset V \times W'$ l'ensemble des points au-dessus desquels le revêtement U' est ramifié; c'est un ensemble algébrique k-fermé, qui ne rencontre pas $V \times \{w\}$ puisque par hypothèse $U_1 = U_w'$ est non ramifié. Il est donc contenu dans un produit $V \times Z'$, où Z' est un sous-ensemble algébrique k-fermé de W'; posons $W = W' - Z'$, et soit U l'image réciproque de $V \times W$ dans U'. Le revêtement U est non ramifié, et l'on a $U_w = U_w' = U_1$; choisissons alors dans W un point a, rationnel sur k. D'après le corollaire 2 au théorème 1 les ensembles algébriques U_w et U_a sont isomorphes. Il s'ensuit que U_a est irréductible, et que c'est un revêtement de V isomorphe à U_1, cfqd.

THÉORÈME 4. *Une variété normale et complète ne possède qu'un nombre fini de revêtements non ramifiés de degré donné.*

Soit V une telle variété. D'après Chow [1], V est image d'une variété

projective V_1, que l'on peut supposer normale, et qui est birationnellement équivalente à V. Tout revêtement non ramifié de V en définit un de V_1, et l'on est ainsi ramené à démontrer le théorème pour V_1. Autrement dit, nous pouvons supposer que V est une variété *projective*. Soit k un corps de définition de V, et soit C_u une *courbe générique* de V (sur k), au sens défini dans [5], § 3; on sait (*loc. cit.*) que si U est un revêtement de V, défini sur k, la restriction de U à C_u est un revêtement irréductible de C_u. Il s'ensuit, par un raisonnement standard, que deux revêtements non isomorphes de V, tous deux définis sur k, ont des restrictions à C_u qui sont également non isomorphes. En prenant pour k un "domaine universel," on voit donc qu'il suffira de démontrer le théorème 4 pour C_u.

Nous supposerons donc, à partir de maintenant, que V est une *courbe* C, non singulière, complète, de genre g. Nous la supposerons plongée dans un espace projectif S_1, et nous désignerons son degré projectif par d. Si $f: U \to C$ est un revêtement non ramifié de C de degré n, le genre g_U de la courbe U est donné par la formule classique:

$$g_U - 1 = n(g-1),$$

et ne dépend donc pas de U. D'après le théorème de Riemann-Roch, on peut plonger birégulièrement U dans un espace projectif S_2 de dimension g_U (donc fixe), le degré projectif de ce plongement étant $d_2 = 2g_U + 1$. Soit $\Gamma \subset S_1 \times S_2$ le graphe de f. Si H_1 et H_2 sont des hyperplans de S_1 et S_2, posons:

$$H = S_1 \times H_2 + H_1 \times S_2;$$

un calcul immédiat montre que $\deg(\Gamma \cdot H) = d_2 + nd$, donc que ce degré ne dépend pas de U. Cela signifie que, si l'on plonge $S_1 \times S_2$ dans un espace projectif S_3 à la manière habituelle (i.e. de telle sorte que H devienne une section hyperplane dans ce nouveau plongement), le degré de Γ dans S_3 sera $d_2 + nd$, donc sera fixe.

Or, d'après la théorie des coordonnées de Chow, il n'existe qu'un nombre *fini* de familles algébriques irréductibles de cycles de $C \times S_2$ ayant un degré donné (dans S_3). Il nous suffit donc de prouver que, si deux revêtements U_1 et U_2 ont des graphes Γ_1 et Γ_2 qui appartiennent à la même famille algébrique F, ces deux revêtements sont isomorphes.

Pour cela, soit Γ un élément générique de la famille F; nous allons d'abord montrer que $\Gamma \to C$ est un revêtement non ramifié de degré n. Tout d'abord, puisque Γ admet pour spécialisation Γ_1, c'est une variété; comme la projection sur C est compatible avec les spécialisations, la relation

$\mathrm{pr}_1(\Gamma_1) = nC$ entraîne $\mathrm{pr}_1(\Gamma) = nC$, donc l'application $\Gamma \to C$ est de degré n. De plus, si Q est un point de C, on peut prolonger la spécialisation $\Gamma \to \Gamma_1$ en une spécialisation $Q \to Q_1$, et $Q \times S_2$ se spécialise en $Q_1 \times S_2$. Comme le cycle $\Gamma_1 \cdot (Q_1 \times S_2)$ est défini, et se compose de n points distincts, il en est de même du cycle $\Gamma \cdot Q \times S_2$). D'après ce qu'on a vu au no. 1, il s'ensuit bien que Γ est un revêtement non ramifié de degré n de la courbe C.

Nous allons maintenant montrer que les revêtements Γ_1 et Γ sont isomorphes; comme le même argument montrera que Γ_2 et Γ sont isomorphes, il en résultera bien que Γ_1 et Γ_2 sont isomorphes, ce qui achèvera la démonstration, en vertu de ce qui a été dit plus haut.

D'après le théorème 3, le revêtement Γ est isomorphe à un revêtement Γ' défini sur le corps de base; on peut supposer Γ' plongé dans $C \times S_2$ comme ci-dessus. Désignons par T le graphe de l'isomorphisme $\Gamma \to \Gamma'$; c'est une sous-variété de $\Gamma \times \Gamma'$, lui-même contenu dans $C \times S_2 \times C \times S_2$. On a évidemment $\mathrm{pr}_{13}(T) = n\Delta$, où Δ est la diagonale de $C \times C$. Etendons maintenant la spécialisation $\Gamma \to \Gamma_1$ en une spécialisation $T \to T_1$; la variété Γ' reste fixe durant cette spécialisation, puisqu'elle est définie sur le corps de base. On a $\mathrm{pr}_{12}(T_1) = \Gamma_1$, $\mathrm{pr}_{34}(T_1) = \Gamma'$ et $\mathrm{pr}_{13}(T_1) = n\Delta$; les deux premières formules montrent que, ou bien T_1 est le graphe d'une correspondance birationnelle entre Γ_1 et Γ', ou bien T_1 est "dégénérée," i.e. égale à $\Gamma_1 \times \{\gamma'\} + \{\gamma_1\} \times \Gamma'$; la troisième formule exclut cette seconde possibilité. On a donc obtenu une correspondance birationnelle entre les revêtements Γ_1 et Γ', commutant avec les projections sur C. Il s'ensuit que Γ_1 et Γ' sont isomorphes, donc aussi Γ_1 et Γ, cqfd.

Remarques.

1) La démonstration du théorème 4 que nous venons de donner utilise de façon essentielle les résultats du no. 2. Inversement, si l'on pouvait démontrer directement le théorème 4 (dans le cas d'une courbe), on retrouverait facilement ces résultats comme corollaires.

2) Soit C une courbe, et soient $P_1, \cdots, P_r$ des points de C en nombre fini. Il est vraisemblable qu'il n'existe qu'un nombre fini de revêtements de C d'un degré n donné, ramifiés seulement aux points P_i, et n'y admettant pas de ramification supérieure (revêtements "tamely ramified"). Une démonstration de ce fait permettrait d'étendre notablement nos résultats.

3) Dans le cas des revêtements abéliens, le théorème 4 peut aussi se démontrer en utilisant le théorème de Néron-Severi (pour les revêtements d'ordre premier à la caractéristique), et la finitude du premier groupe de

cohomologie (pour les revêtements d'ordre une puissance de la caractéristique). Cf. [7], nos. 15 et 16.

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LIE GROUPS AND LIE HYPERALGEBRAS OVER A FIELD OF CHARACTERISTIC $p > 0$ (VI).*

By JEAN DIEUDONNÉ.

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1. Introduction. In the preceding papers of this series ([8] to [12])[1] our main task had been the study of *abelian* formal Lie groups, and the general theory had not been developed beyond the most elementary theorems. On the contrary, this paper is chiefly devoted to *noncommutative* formal Lie groups. There seems to be little hope at present of using the hyperalgebras, in that part of the theory, to the same effect as they were used in the study of abelian groups [11]. Here the hyperalgebra recedes in the background and becomes a mere technical tool, and the method of attack is based on an entirely different idea, which consists in considering the homomorphisms of a formal Lie group G into *linear groups*. When G has *no center*, there exist such homomorphisms which are injective; when such homomorphisms exist (in which case G is said to be *representable*), it is possible to define an "algebraic hull" of the image of G by such a homomorphism, following the pattern of the corresponding techniques used by C. Chevalley for Lie algebras [6] and by A. Borel for arbitrary (non formal) subgroups of linear groups [2]. I must record here my indebtedness to these two fundamental papers: not only do they suggest, by analogy, the results to be expected for

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[1] I take this opportunity to correct a few misprints and minor errors in these papers: in [9], p. 237, formula (45), the summation should extend over the range $0 < \mu < \lambda$. In [10], p. 64, in lines 1-2 from top, the condition "et n'ont aucune composante d'indice $< i$ qui soit non nulle" should be omitted, as well as the statement "toutes les composantes d'indice $< i$ sont nulles" in lines 12-13 from top. On the same page, the index (or exponent) i in G_i, $\bar{G}_i$, $\mathfrak{m}_i$, $\bar{\mathfrak{m}}_i$, $u^{(i)}$, $\bar{u}^{(i)}$ should be replaced by $i+1$. On p. 69 of the same paper, equation (39) should read $\Delta(\mathrm{Hex}(x_0, x_1, \ldots)) = c \cdot \mathrm{Hex}(x_0, x_1, \ldots)$ where c is a constant, and in line 10 from bottom, y should be replaced by x. In [12,] p. 208, line 9 from top, read $<$ instead of $\leqq$; p. 210, line 9 from top, read $\xi^{p^{-1}}$; p. 211, lines 3 and 6 from bottom, replace r by $r+1$; p. 219, line 12 from bottom, there should be a bar over the second $\mathfrak{r}_{m+1}$; p. 221, line 4 from top, there should be a bar over $\mathfrak{q}_{m+1,r}$; p. 222, line 14 from bottom, replace p^r by p^{r+1}; p. 238, formula (65), read $E(\mathbf{Y}_1^{(q,2)})$ instead of $(\mathbf{Y}_1^{(q,2)})$. In [13], p. 252, formula (3), the summation is over the index α; p. 254, line 18 from top, X_{hi} should be replaced by $\bar{X}_{hi}$. P. 255, line 3 from bottom, p. 256, line 10 from top, p. 257, line 18 from bottom, replace the exponent β_{k-1} by $\beta_k - 1$.

formal Lie groups, and sometimes the methods of proof by which they can be obtained, but it turns out that a surprisingly large number of the theorems of A. Borel for *algebraic* groups can be used to prove their counterparts for *formal* Lie groups (even when there does not seem to be any possibility of extending their *method* of proof): we mention in particular many of Borel's results on solvable maximal subgroups, maximal tori and Cartan subgroups. This is mainly due to a fundamental result (suggested by a similar theorem of Chevalley on Lie algebras in characteristic 0) according to which the *derived group* of a representable formal Lie group is essentially *algebraic*. From this it follows also that *simple* formal Lie groups which are not abelian are essentially *algebraic*, and the latter have recently been determined by C. Chevalley [7], who has shown that they are the same as in characteristic 0, namely the groups of the Killing-Cartan classification. To sum up, it seems that a general formal Lie group is essentially built up out of *algebraic* groups and *abelian* formal Lie groups; and, contrary to what might have been expected, the theory of non-commutative formal Lie groups does not exhibit startlingly new features when compared to the classical theory; the radically new phenomena (should one say the "pathology" because they have never been met before?) occur in the theory of *abelian* Lie groups (and in the general case, are confined to the *center* of the group).

There is one additional point, however, which should be duly emphasized: in order to develop the method outlined above, it is first necessary to have at hand the usual tools of group theory, such as subgroups, quotient groups, group of commutators, etc. This might seem at first glance to offer no problem at all, but one must remember that the *set-theoretic* background, which is usually taken for granted (and plays an essential part in the development of the papers [2] and [6] quoted above), simply does not exist in our theory: a "formal Lie group" *is not* a set, has no "elements" nor "subsets", and we are thus denied the use of set-theoretic arguments, at least until we have been able to find substitutes for them by defining anew all the notions which, in the theory of algebraic groups or of Lie algebras, are based on the theory of sets. This is the object of the first two parts of the paper, part III being devoted to the proof of the main results. It is in the definition of these basic notions that the hyperalgebra is particularly useful, for there the set-theoretic notions again become available; however there arise unexpected difficulties, and actually the hardest theorem to prove is that the subgroups of a formal Lie group form a *lattice* (for a convenient definition of "inclusion"), a fact which would hardly need a proof in the classical theories.

I am much indebted to M. Rosenlicht for useful information on algebraic geometry and for many stimulating comments.

2. Preliminaries. We do not recall here the definitions of a formal Lie group G of finite dimension n over a field K, of the hyperalgebra $\mathfrak{G}$ of such a group,[2] of a homomorphism $\boldsymbol{u}$ of G into a formal Lie group G_1 (over the same field K) and of the derived homomorphism $\boldsymbol{u}'$ of $\mathfrak{G}$ into the hyperalgebra $\mathfrak{G}_1$ of G_1 (see [8]); until no. 14, when we speak of a "group", a formal Lie group will always be meant. It will be convenient, when considering a system $\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$ of n indeterminates which intervene in the definition of G, to refer to it as a "generic point" of G; when dealing with several "generic points", $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \cdots$, to say that they are "independent" will mean that all the indeterminates in these various systems are distinct from one another. The notation of these systems by boldface characters will prevent any confusion with the usual notion of "generic point". We recall that $\boldsymbol{x}\cdot\boldsymbol{y}$ or $\boldsymbol{xy}$ is an abbreviation for the system $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y})=(\phi_1(\boldsymbol{x},\boldsymbol{y}), \cdots, \phi_n(\boldsymbol{x},\boldsymbol{y}))$ of n formal power series which defines the group law of G; for any system $\gamma=(\gamma_1, \cdots, \gamma_n)$ of n integers $\geqq 0$, $\boldsymbol{x}^\gamma$ is the monomial $x_1^{\gamma_1}\cdots x_n^{\gamma_n}$; if $\boldsymbol{u}=(u_1, \cdots, u_n)$, $\boldsymbol{v}=(v_1, \cdots, v_n)$ are two systems of n power series (without constant terms) in certain indeterminates, $\boldsymbol{uv}$ and $\boldsymbol{u}^\gamma$ will then designate the system of power series (resp. the power series) obtained by substituting u_i for x_i and v_i for y_i $(1\leqq i\leqq n)$ in $\boldsymbol{xy}$ and $\boldsymbol{x}^\gamma$ respectively.

A formal Lie group G over a field K can also be considered as a formal Lie group over any extension L of K; when we do so, we will write it $G_{(L)}$; the necessity of distinguishing these two groups arises from the fact that the notions of homomorphism and of hyperalgebra depend on the ground field. More precisely, any homomorphism of G into $\bar{G}$ is also a homomorphism of $G_{(L)}$ into $\bar{G}_{(L)}$, but not conversely; and the hyperalgebra of $G_{(L)}$ is (as an algebra) obtained by extension to L of the field of scalars of $\mathfrak{G}$, and will accordingly be written $\mathfrak{G}_{(L)}$. We will only consider in this paper the case in which the ground field K is a *perfect* field of characteristic $p>0$.

We keep the notations of [8], except that we write $\mathfrak{s}_{r-1}$ instead of $\mathfrak{s}_r$ for the algebra of left-invariant special semi-derivations of height $\leqq r$; $\mathfrak{s}_{r-1}$ is thus the enveloping algebra of the Lie p-algebra $\mathfrak{g}_{r-1}$ of left-invariant semi-derivations of height $\leqq r-1$. We recall that the identity I and the left-invariant differential operators Z_α (defined by the condition that $Z_\alpha(\boldsymbol{e})=D_\alpha(\boldsymbol{e})$) form a basis for the hyperalgebra $\mathfrak{G}$ over K, which we will call the

[2] I will systematically use the convention by which a German capital represents the hyperalgebra of a group represented by the corresponding Latin capital.

structural basis of $\mathfrak{G}$; we write $Z_0 = I$, and we denote by $\mathfrak{G}^+$ the two-sided ideal of $\mathfrak{G}$ having as a basis the Z_α corresponding to indices $\alpha \neq 0$. We write as usual ϵ_i for the system of indices $(0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 at the i-th place, and we put $X_{hi} = Z_{p^h \epsilon_i}$; if $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\alpha_i = \sum_{h=0}^{\infty} \alpha_{hi} p^h$ (p-adic development of the α_i), X_α will denote the product of the $X_{hi}^{\alpha_{hi}}$ taken in an order which may vary according to the problem which is considered (and in particular may be different from the order chosen in [8]); once that order is chosen, the X_α form a basis for $\mathfrak{G}$, and the X_α such that $\alpha \neq 0$ a basis for $\mathfrak{G}^+$.

We recall that there is an injective homomorphism $X \to X^0$ (for the structure of algebra) of $\mathfrak{G}$ into the algebra $\mathfrak{G} \otimes \mathfrak{G}$, such that I^0 is the identity $I \otimes I$ and $Z^0_\alpha = \sum_{0 \leq \beta \leq \alpha} Z_\beta \otimes Z_{\alpha-\beta}$ for any $\alpha \neq 0$; we will say that this isomorphism is the *structural mapping* of $\mathfrak{G}$ into $\mathfrak{G} \otimes \mathfrak{G}$.

Finally, for any (positive or negative) integer s, we will denote by $G^{(s)}$ the formal Lie group obtained by raising to the p^s-th power the coefficients of the series defining the group law of G. The "Frobenius homomorphism" $\boldsymbol{p}$ is a homomorphism of $G^{(s)}$ into $G^{(s+1)}$ such that $p_i(\boldsymbol{x}) = x^p_i$ for $1 \leq i \leq n$. The derived homomorphism $\boldsymbol{p}'$ (also called the "Frobenius homomorphism") can be considered as a semi-linear mapping of the hyperalgebra $\mathfrak{G}$ into itself, corresponding to the automorphism $\xi \to \xi^{p^{-1}}$ of K, and is such that $\boldsymbol{p}'(X_{hi}) = X_{h-1,i}$ for each pair (h, i) (with $X_{-1,i} = 0$ by convention); the kernel $\boldsymbol{p}'^{-1}(0)$ is thus the set of all linear combinations of the X_α which contain at least one of the X_{0i}.

3. Pseudo-canonical group laws. In [12], we have defined the notion of *free Lie hyperalgebra* corresponding to formal Lie groups of dimension n; such a hyperalgebra $\mathfrak{H}_n(K)$ is not uniquely defined, but we have proved in [12, no. 19] that it is always possible to define its structural basis (V_α) (where the $\alpha = (\alpha_\rho)$ $(\rho \in R)$ correspond to an infinite set of indices R, containing the interval $1 \leq i \leq n$) such that when $\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \cdots + \alpha_n \epsilon_n$ is in $\boldsymbol{N}^n$, we have $V_\alpha = V_{\alpha_1 \epsilon_1} V_{\alpha_2 \epsilon_2} \cdots V_{\alpha_n \epsilon_n}$, and moreover that, for each i $(1 \leq i \leq n)$ the $V_{\alpha_i \epsilon_i}$ (α_i integer ≥ 0) are (noncommutative) polynomials in the U_{hi} *alone* ($h \geq 0$ variable). The free hyperalgebra $\mathfrak{H}_n(K)$ being fixed once for all and satisfying the preceding condition, we will say that the group law of a group G of dimension n is *pseudo-canonical* if there exists a homomorphism $\boldsymbol{u}'$ of $\mathfrak{H}_n(K)$ onto $\mathfrak{G}$ such that $Z_\alpha = \boldsymbol{u}'(V_\alpha)$ for each $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_n \epsilon_n \in \boldsymbol{N}^n$. The way in which $\mathfrak{H}_n(K)$ has been defined proves that

LEMMA 1. *If the group law of G is pseudo-canonical, then, for every $\alpha=\alpha_1\epsilon_1+\cdots+\alpha_n\epsilon_n\in \boldsymbol{N}^n$, we have $Z_\alpha=Z_{\alpha_1\epsilon_1}Z_{\alpha_2\epsilon_2}\cdots Z_{\alpha_n\epsilon_n}$; furthermore, if $p^h\leqq\alpha_i<p^{h+1}$, $Z_{\alpha_i\epsilon_i}$ is an element of the subalgebra of $\mathfrak{G}$ generated by $X_{0i}, X_{1i},\cdots, X_{hi}$.*

Moreover, we have shown in [12, no. 12, th. 2] that any group G of dimension n is isomorphic to a group $\bar{G}$ with pseudo-canonical group law. Actually, the proof of that result yields a "standard process" for the construction of such an isomorphism, which we summarize for future use: starting with the group $G_0=G$, one defines a sequence (G_r) of groups, and for each r an isomorphism $\boldsymbol{u}_r$ of G_r onto G_{r+1}, such that the following conditions are satisfied. Let (Z_α) be the structural basis of $\mathfrak{G}_r$, and let $\boldsymbol{f}$ be the algebra homomorphism of $\mathfrak{H}_n(K)$ onto $\mathfrak{G}_r$ such that $\boldsymbol{f}(U_{hi})=X_{hi}$ for all pairs (h,i) $(h\geqq 0, 1\leqq i\leqq n)$; then we have $\boldsymbol{f}(V_\alpha)=Z_\alpha$ for all indices $\alpha=\alpha_1\epsilon_1+\cdots+\alpha_n\epsilon_n$ of height $<r$. In general, if we write $\boldsymbol{f}(V_\alpha)=\sum_\lambda b_{\alpha\lambda}Z_\lambda$, we thus see that we have $b_{\alpha\alpha}=1$ and $b_{\alpha\lambda}=0$ for $\lambda\neq\alpha$, *for any $\alpha\in\boldsymbol{N}^n$ such that $h(\alpha)<r$ and for $\alpha=p^r\epsilon_i$* $(1\leqq i\leqq n)$. Now, if $\boldsymbol{x}'=(x'_i)$ is a generic point for G_{r+1}, $\boldsymbol{u}^{-1}{}_r$ is defined by the formulae

$$u^{-1}{}_{ri}(\boldsymbol{x}')=\sum_\alpha b_{\alpha\epsilon_i}\boldsymbol{x}'^{\alpha} \qquad (1\leqq i\leqq n)$$

and the preceding remark shows that, on the right-hand side, we have the term x'_i, all the other terms have a height $\geqq r$, and the monomials $x'^{p^r}_j$ have coefficient 0 $(1\leqq j\leqq n)$. From which it follows that

$$u_{ri}(\boldsymbol{x})=x_i+v_{ri}(\boldsymbol{x}) \qquad (1\leqq i\leqq n) \tag{1}$$

where, in the power series v_{ri} $(1\leqq i\leqq n)$, all monomials *have a height* $\geqq r$, and there are no monomials $x^{p^r}_j$ $(1\leqq j\leqq n)$. Having thus defined the isomorphism $\boldsymbol{u}_r$, it follows from the preceding remark that the "infinite product" $\cdots\boldsymbol{u}_r\boldsymbol{u}_{r-1}\cdots\boldsymbol{u}_0$ is meaningful and is an isomorphism of G onto a group $\bar{G}$ with pseudo-canonical group law.

I. Homomorphisms and Subgroups.

4. Homomorphisms. One of our main tools in this paper will be the *homomorphism theorem* [8, th. 6, p. 107]. We recall that, according to that theorem (in which the assumption that K is *perfect* is an essential one), if $\boldsymbol{u}$ is a homomorphism of G into G', it is possible to "change variables" in G and G' (in other words, to replace $\boldsymbol{u}$ by $\boldsymbol{guf}$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are isomorphisms) in such a way that

$$
\begin{aligned}
&u_i(x) = x_i \quad \text{for} \quad 1 \leqq i \leqq r_0 \\
&u_i(x) = x^p{}_i \quad \text{for} \quad r_0 + 1 \leqq i \leqq r_0 + r_1 \\
&\qquad \cdots\cdots\cdots\cdots \\
&u_i(x) = x^{p^t}{}_i \quad \text{for} \quad r_0 + \cdots + r_{t-1} + 1 \leqq i \leqq r_0 + \cdots + r_t = \rho \\
&u_i(x) = 0 \quad \text{for} \quad i > \rho.
\end{aligned} \tag{2}
$$

To these formulae correspond, for the derived homomorphism $\boldsymbol{u}'$, the formulae

$$
\begin{aligned}
&\boldsymbol{u}'(X_{hi}) = X'_{hi} \quad \text{for} \quad 1 \leqq i \leqq r_0 \\
&\boldsymbol{u}'(X_{hi}) = X'_{h-1,i} \quad \text{for} \quad r_0 + 1 \leqq i \leqq r_0 + r_1 \\
&\qquad \cdots\cdots\cdots\cdots \\
&\boldsymbol{u}'(X_{hi}) = X'_{0i} \quad \text{for} \quad r_0 + \cdots + r_{h-1} + 1 \leqq i \leqq r_0 + \cdots + r_h \\
&\boldsymbol{u}'(X_{hi}) = 0 \quad \text{for} \quad i > r_0 + \cdots + r_h
\end{aligned} \tag{3}
$$

where we agree to put $r_h = 0$ for $h > t$. The integers r_k (some of which may be 0), which we will call the *partial ranks* of $\boldsymbol{u}$, actually depend only on the original homomorphism $\boldsymbol{u}$ and not on the particular isomorphisms $\boldsymbol{f}$, $\boldsymbol{g}$ which yield the formulae (2); $\rho = r_0 + \cdots + r_t$ is the *rank* of $\boldsymbol{u}$, and t (greatest integer for which $r_t \neq 0$) the *height* of $\boldsymbol{u}$. We say that the homomorphism $\boldsymbol{u}$ is *separable* if $t = 0$, *inseparable* if $t > 0$, *trivially inseparable* if $t > 0$, $\rho = r_t$; in that case, $\boldsymbol{u}$ can be written $\boldsymbol{v} \circ \boldsymbol{p}^t$, where $\boldsymbol{v}$ is a separable homomorphism of $G^{(t)}$ into G'. We say that $\boldsymbol{u}$ is *injective* if $\rho = \dim G$, *surjective* if $\rho = \dim G'$; a *monomorphism* (resp. an *epimorphism*) is a homomorphism which is both injective (resp. surjective) *and separable*. From (3) we derive at once the following criterion:

PROPOSITION 1. *In order that a homomorphism $\boldsymbol{u}$ of G into G' be separable, a necessary and sufficient condition is that the image $\boldsymbol{u}'(\mathfrak{g}_0)$ of the Lie algebra of G by the derived homomorphism $\boldsymbol{u}'$ be a Lie subalgebra of dimension ρ of the Lie algebra of G'.*

COROLLARY. *If $\boldsymbol{u}$, $\boldsymbol{v}$ are both monomorphisms* (resp. *both epimorphisms*) *of G into G' and of G' into G'' respectively, then $\boldsymbol{v} \circ \boldsymbol{u}$ is also a monomorphism* (resp. *an epimorphism*).

However, we will see later (no. 21) that $\boldsymbol{v} \circ \boldsymbol{u}$ may well be inseparable even if $\boldsymbol{u}$ and $\boldsymbol{v}$ are separable homomorphisms.

Following M. Rosenlicht [16], we will say that a homomorphism $\boldsymbol{u}$ is an *isogeny* if it is both injective and surjective; an isomorphism is a separable isogeny; an isogeny which is trivially inseparable is obtained by composing an isomorphism with an iterate of the Frobenius homomorphism.

PROPOSITION 2. *If $\boldsymbol{u}$ is an isogeny of height t of G onto G', there is a unique isogeny $\boldsymbol{v}$ of G' onto $G^{(t)}$ such that $\boldsymbol{v}\circ\boldsymbol{u}=\boldsymbol{p}^t$.*

We can suppose $\boldsymbol{u}$ is given by (2) with $\rho=n=\dim G=\dim G'$. Let $\phi=(\phi_i)$ and $\psi=(\psi_i)$ be the group laws of G and G' respectively; by definition, we have, for two independent generic points $\boldsymbol{x}=(x_i)$, $\boldsymbol{y}=(y_i)$ of G, the following identities:

$$\begin{aligned}(4)\qquad &\psi_i(x_1,\cdots,x_{r_0},x^p{}_{r_0+1},\cdots,x^{p^2}{}_{r_0+r_1+1},\cdots,x^{p^t}{}_n;\\ &y_1,\cdots,y_{r_0},y^p{}_{r_0+1},\cdots,y^{p^t}{}_n)=(\phi_i(x_1,\cdots,x_n;y_1,\cdots,y_n))^{p^h}\end{aligned}$$

for $r_0+\cdots+r_{h-1}+1\leqq i\leqq r_0+\cdots+r_h$, $0\leqq h\leqq t$. Raise both sides of (4) to the power p^{t-h}, and denote by $\phi^{(t)}=(\phi^{(t)}{}_i)$ the group law of $G^{(t)}$; then if we put $x'_i=x^{p^h}{}_i$ for $r_0+\cdots+r_{h-1}+1\leqq i\leqq r_0+\cdots+r_h$, $0\leqq h\leqq t$, the resulting identities can be written

$$\begin{aligned}&\phi_i^{(t)}(x'_1{}^{p^t},\cdots,x'^{p^t}{}_{r_0},x'^{p^{t-1}}{}_{r_0+1},\cdots,x'^{p^{t-1}}{}_{r_0+r_1},x'^{p^{t-2}}{}_{r_0+r_1+1},\cdots,x'_n;\\ &y'_1{}^{p^t},\cdots,y'^{p^t}{}_{r_0},y'^{p^{t-1}}{}_{r_0+1},\cdots,y'_n)=(\psi_i(x'_1,\cdots,x'_n;y'_1,\cdots,y'_n))^{p^{t-h}}\end{aligned}$$

for $r_0+\cdots,+r_{h-1}+1\leqq i\leqq r_0+\cdots+r_h$, $0\leqq h\leqq t$. If we define $v_i(\boldsymbol{x}')$ to be equal to $x'^{p^{t-h}}{}_i$ for $r_0+\cdots+r_{h-1}+1\leqq i\leqq r_0+\cdots+r_h$, $0\leqq h\leqq t$, this shows that $\boldsymbol{v}=(v_i)$ is an isogeny which satisfies the required conditions, and it is obviously unique.

We will say that two groups G, G' are *isogenous* if there exists an isogeny of G onto a group $G'^{(s)}$, where s is a positive or negative integer. This relation is clearly *transitive*, for if $\boldsymbol{u}=(u_i)$ is an isogeny of G onto G', and if $\boldsymbol{u}^{(s)}$ denotes the system of series $u_i{}^{(s)}$ obtained by raising the coefficients of the u_i to the power p^s, $\boldsymbol{u}^{(s)}$ is clearly an isogeny of $G^{(s)}$ onto $G'^{(s)}$. Moreover Proposition 2 shows that if G is isogenous to G', G' is isogenous to G, for if there is an isogeny of height t of G onto $G'^{(s)}$, there is an isogeny of G' onto $G^{(t-s)}$. In other words, isogeny is an *equivalence relation* between groups of same dimension. It is easy to see that that relation can be defined in the same terms as those used by M. Rosenlicht [16, p. 147] to define isogeny between algebraic groups: namely, G_1 and G_2 are isogenous if and only if there exists a group G_3 and two isogenies of G_3 on G_1 and G_2 respectively. For if there is an isogeny $\boldsymbol{u}$ of G_1 onto $G_2{}^{(s)}$, then we can suppose $s\geqq 0$ (otherwise $\boldsymbol{p}^{-s}\circ\boldsymbol{u}$ is an isogeny of G_1 onto G_2), and then take $G_3=G_1{}^{(-s)}$, the isogenies on G_1 and G_2 being $\boldsymbol{p}^s$ and $\boldsymbol{u}^{(-s)}$ respectively. Conversely, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are isogenies of G_3 on G_1 and G_2, respectively, then by Proposition 2, there is an isogeny $\boldsymbol{u}$ of G_1 onto some $G_3{}^{(t)}$, and $\boldsymbol{w}\circ\boldsymbol{u}^{(t)}$ is an isogeny of G_1 onto $G_2{}^{(t)}$.

5. Subgroups and subhyperalgebras. We slightly modify the formulation of the definition of a subgroup given in [8]: we define a *subgroup of dimension* m of a formal Lie group G of dimension n as a pair $(H, \boldsymbol{f})$ consisting of a formal Lie group H of dimension m and a *monomorphism* $\boldsymbol{f}$ of H into G. That this definition is essentially the same as that of [8] is seen by remarking that it follows from the homomorphism theorem that there is an isomorphism $\boldsymbol{v}$ of G onto a group $\bar{G}$ and an isomorphism $\boldsymbol{u}$ of a group $\bar{H}$ onto H such that $\boldsymbol{v} \circ \boldsymbol{f} \circ \boldsymbol{u}$ will have the form (2) with $\rho = r_0 = m$; we may say in that case that the subgroup $(\bar{H}, \boldsymbol{v} \circ \boldsymbol{f} \circ \boldsymbol{u})$ of $\bar{G}$ is *typical*, and is *defined by the equations* $x_{m+1} = \cdots = x_n = 0$.

Moreover, we shall *identify* two subgroups $(H_1, \boldsymbol{f}_1)$, $(H_2, \boldsymbol{f}_2)$ of G if there is an isomorphism $\boldsymbol{u}$ of H_1 onto H_2 such that $\boldsymbol{f}_1 = \boldsymbol{f}_2 \circ \boldsymbol{u}$[3]; and we will say that the subgroup $(H_1, \boldsymbol{f}_1)$ is *contained* in the subgroup $(H_2, \boldsymbol{f}_2)$ if there is a monomorphism $\boldsymbol{u}$ of H_1 into H_2 such that $\boldsymbol{f}_1 = \boldsymbol{f}_2 \circ \boldsymbol{u}$.

PROPOSITION 3. *In order that a subgroup $(H_1, \boldsymbol{f}_1)$ of G be contained in a subgroup $(H_2, \boldsymbol{f}_2)$, a necessary and sufficient condition is that if $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{G}$ are the hyperalgebras of H_1, H_2 and G, the subalgebra $\boldsymbol{f}'_1(\mathfrak{H}_1)$ of $\mathfrak{G}$ be contained in the subalgebra $\boldsymbol{f}'_2(\mathfrak{H}_2)$.*

The condition is obviously necessary. Conversely, suppose it is satisfied, and observe that, from the homomorphism theorem, it follows that a monomorphism $\boldsymbol{u}$ is *characterized* by the fact that $\boldsymbol{u}'$ is *injective*: $\boldsymbol{f}'_2$ can therefore be considered as an isomorphism of $\mathfrak{H}_2$ onto the subalgebra $\boldsymbol{f}'_2(\mathfrak{H}_2)$ of $\mathfrak{G}$, and in addition we know [9, p. 221] that $(\boldsymbol{f}'_2(Y))^0 = (\boldsymbol{f}'_2 \otimes \boldsymbol{f}'_2)(Y^0)$ for every $Y \in \mathfrak{H}_2$; if $\boldsymbol{s}_2$ is the isomorphism of $\boldsymbol{f}'_2(\mathfrak{H}_2)$ onto $\mathfrak{H}_2$, inverse to $\boldsymbol{f}'_2$, we have therefore $(\boldsymbol{s}_2(Z))^0 = (\boldsymbol{s}_2 \otimes \boldsymbol{s}_2)(Z^0)$ for every $Z \in \boldsymbol{f}'_2(\mathfrak{H}_2)$. Let $\boldsymbol{r} = \boldsymbol{s}_2 \circ \boldsymbol{f}'_1$; this is an injective homomorphism of $\mathfrak{H}_1$ into $\mathfrak{H}_2$ such that $(\boldsymbol{r}(X))^0 = (\boldsymbol{r} \otimes \boldsymbol{r})(X^0)$ for every $X \in \mathfrak{H}_1$. But that condition implies that $\boldsymbol{r}$ is the derived homomorphism of a monomorphism $\boldsymbol{u}$ of H_1 into H_2 [10, p. 59, prop. 2], and as $\boldsymbol{f}'_1 = \boldsymbol{f}'_2 \circ \boldsymbol{r}$ by definition, we have $\boldsymbol{f}_1 = \boldsymbol{f}_2 \circ \boldsymbol{u}$.

For a subgroup $(H, \boldsymbol{f})$ of G, we will say that $\boldsymbol{f}'(\mathfrak{H})$ is the *subhyperalgebra* of $\mathfrak{G}$ corresponding to the subgroup; Proposition 3 shows that the subhyperalgebras of $\mathfrak{G}$ thus correspond in a one-to-one manner to subgroups of G. We will now characterize them independently of their origin as images of hyperalgebras:

[3] For any formal Lie group G' isomorphic to G and any isomorphism $\boldsymbol{f}$ of G' onto G, the subgroup $(G', \boldsymbol{f})$ is thus identified to the subgroup $(G, 1)$ of G (where 1 designates the identity isomorphism, defined by the n series x_i $(1 \leqq i \leqq n)$).

THEOREM 1. *In order that a subalgebra $\mathfrak{H}$ of the hyperalgebra $\mathfrak{G}$ be a subhyperalgebra, the following conditions are necessary and sufficient:*

1°. *For any $r \geqq 0$ and any $X \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$, $X^0 - 1 \otimes X - X \otimes 1$ belongs to $(\mathfrak{z}_r \cap \mathfrak{H}) \otimes (\mathfrak{z}_r \cap \mathfrak{H})$.*

2°. *For any $r \geqq 0$, the algebra $\mathfrak{z}_r \cap \mathfrak{H}$ is generated by the Lie p-algebra $\mathfrak{g}_r \cap \mathfrak{H}$.*[4].

3°. *For any element $X \in \mathfrak{g}_r \cap \mathfrak{H}$, there is an element $Y \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$ such that $X = \boldsymbol{p}'(Y)$.*

It is immediate to see that the conditions are necessary, by considering a *typical* subgroup of G. To prove the conditions are sufficient, consider the Lie p-algebra $\mathfrak{g}_0 \cap \mathfrak{H}$; performing if necessary a linear transformation on the indeterminates x_i, we can suppose that the structural basis of $\mathfrak{G}$ is such that the X_{0i} of index $\leqq m$ constitute a basis for $\mathfrak{g}_0 \cap \mathfrak{H}$; we denote by J the interval $1 \leqq i \leqq m$ in $\boldsymbol{N}$. In order to build up a proof by induction, let us first suppose that G satisfies the following assumption:

(A_r) *The group law of G is pseudo-canonical* (no. 3), *and the elements X_α such that $\alpha \in \boldsymbol{N}^J$ and $h(\alpha) \leqq r$ form a basis for $\mathfrak{z}_r \cap \mathfrak{H}$.*

Observe that, owing to Lemma 1, the Z_α such that $\alpha \in \boldsymbol{N}^J$ and $h(\alpha) \leqq r$ also form a basis for $\mathfrak{z}_r \cap \mathfrak{H}$.

From assumption 3°, for every index $i \in J$, there exists in $\mathfrak{g}_{r+1} \cap \mathfrak{H}$ an element $Y_{r+1,i}$ such that $\boldsymbol{p}'(Y_{r+1,i}) = X_{ri}$, and we can obviously write $Y_{r+1,i} = X_{r+1,i} + \sum_\lambda c_{r\lambda} Z_\lambda$, where the summation extends over all indices λ of height $h(\lambda) \leqq r$. I claim that $c_{r\lambda} = 0$ for all λ which are not in $\boldsymbol{N}^J$ and are different from the ϵ_j $(1 \leqq j \leqq n)$. Indeed, if we form the expression $Y^0_{r+1,i} - 1 \otimes Y_{r+1,i} - Y_{r+1,i} \otimes 1$, which belongs to $(\mathfrak{z}_r \cap \mathfrak{H}) \otimes (\mathfrak{z}_r \cap \mathfrak{H})$ by assumption 1°, we obtain

$$\sum_{0<h<p^{r+1}} Z_{he_i} \otimes Z_{(p^{r+1}-h)\epsilon_i} + \sum_\lambda c_{r\lambda} \Big(\sum_{0<\mu<\lambda} Z_\mu \otimes Z_{\lambda-\mu} \Big)$$

and all the terms $Z_\gamma \otimes Z_\delta$ which intervene in that sum are distinct, hence linearly independent. But from assumption (A_r) it follows that in the preceding sum the only terms $Z_\gamma \otimes Z_\delta$ which can have a coefficient $\neq 0$ are those for which *both* γ and δ belong to $\boldsymbol{N}^J$, whence our assertion.

We can therefore suppose, by subtracting if necessary from $Y_{r+1,i}$ an element of $\mathfrak{z}_r \cap \mathfrak{H}$ (and remembering that by definition $\boldsymbol{p}'(Z_\lambda) = 0$ for all

[4] I have not been able to determine if condition 1° implies condition 2° or not.

the Z_λ which intervene in $Y_{r+1,i}$), that $Y_{r+1,i} = X_{r+1,i} + \sum_{j=1}^{n} a_{ij} X_{0j}$ $(1 \leqq i \leqq m)$. Consider now an isomorphism $\boldsymbol{u}$ of G onto a group G' such that we have (writing Z'_α and X'_{hi} for the elements of the structural basis of $\mathfrak{G}'$)

$$\begin{aligned} &\boldsymbol{u}'(X_{hi}) = X'_{hi} \text{ for } h \leqq r \text{ and for } h = r+1, i \notin J; \\ &\boldsymbol{u}'(Y_{r+1,i}) = X'_{r+1,i} \text{ for } i \in J. \end{aligned} \tag{5}$$

This can be achieved by taking for instance

$$u_i(\boldsymbol{x}) = x_i - \sum_{j=1}^{m} a_{ji} x_j^{p^{r+1}} \qquad \text{for } 1 \leqq i \leqq n \tag{6}$$

as it is readily verified. By assumption, there is a homomorphism $\boldsymbol{f}$ of $\mathfrak{H}_n(K)$ onto $\mathfrak{G}$ such that $\boldsymbol{f}(V_\alpha) = Z_\alpha$ for every index; from (5) it follows that we have $\boldsymbol{u}'(\boldsymbol{f}(V_\alpha)) = Z'_\alpha$ for all indices α *of height* $\leqq r$. If we denote by $\boldsymbol{v}$ the isomorphism of G' onto a group G'' obtained by the "standard process" of no. 3, it follows at once from the description of that process and from the preceding remark that we can write

$$v_i(\boldsymbol{x}') = x'_i + w_i(\boldsymbol{x}') \qquad (1 \leqq i \leqq n) \tag{7}$$

where in the series w_i, all monomials *have a height* $\geqq r+1$, and there are no monomials $x'^{p^{r+1}}_j$ $(1 \leqq j \leqq n)$. Writing X''_{hi} for the elements of the structural basis of $\mathfrak{G}''$, we deduce therefore from (5) that

$$\begin{aligned} &\boldsymbol{v}'(\boldsymbol{u}'(X_{hi})) = X''_{hi} \text{ for } h \leqq r \text{ and for } h = r+1, i \notin J; \\ &\boldsymbol{v}'(\boldsymbol{u}'(Y_{r+1,i})) = X''_{r+1,i} \text{ for } i \in J. \end{aligned}$$

Replacing (for the moment) G by G'', we can therefore suppose from now on that in G the group law is pseudo-canonical, and that we have $Y_{r+1,i} = X_{r+1,i}$ for $i \in J$.

Next we remark that $\mathfrak{g}_{r+1} \cap \mathfrak{H}$ has a basis consisting of the Z_α of height $\leqq r$ such that $\alpha \in \boldsymbol{N}^J$, and of the m elements $X_{r+1,i}$ $(i \in J)$. Indeed, if we consider an arbitrary element $Y = \sum_{i=1}^{n} b_i X_{r+1,i} + \sum_{\lambda} c_\lambda Z_\lambda$ of $\mathfrak{g}_{r+1} \cap \mathfrak{H}$ (summation extended over all the indices λ of height $\leqq r$), and if we express that $Y^0 - 1 \otimes Y - Y \otimes 1$ belongs to $(\mathfrak{s}_r \cap \mathfrak{H}) \otimes (\mathfrak{s}_r \cap \mathfrak{H})$, we see at once (using assumption (A_r)) that the only coefficients which can be $\neq 0$ are the b_i for $i \in J$ and the c_λ for which $\lambda \in \boldsymbol{N}^J$. As $\mathfrak{g}_{r+1} \cap \mathfrak{H}$ is a Lie p-algebra, the elements $X^{p}_{r+1,i}$ and $[X_{r+1,i}, X_{r+1,j}]$ $(i, j$ in $J)$ are in $\mathfrak{g}_{r+1} \cap \mathfrak{H}$; using now assumption 2°, we see that the elements X_α of height $\leqq r+1$ and such that $\alpha \in \boldsymbol{N}^J$, form a basis for $\mathfrak{s}_{r+1} \cap \mathfrak{H}$. Due to the fact that the group law is pseudo-canonical,

Lemma 1 shows that the Z_α such that $h(\alpha) \leqq r+1$ and $\alpha \in \boldsymbol{N}^J$ also form a basis for $\mathfrak{s}_{r+1} \cap \mathfrak{H}$.

Summing up what we have done, and slightly changing the notations, we have proved the following result: if G_r is a group, in the hyperalgebra $\mathfrak{G}_r$ of which the subalgebra $\mathfrak{H}_r$ verifies the assumptions of Theorem 1 and the assumptions (A_r), then there is an isomorphism $\boldsymbol{g}_r$ mapping G_r onto a group G_{r+1} in such a way that $\mathfrak{H}_{r+1} = \boldsymbol{g}'_r(\mathfrak{H}_r)$ now satisfies the assumptions of Theorem 1 and the assumptions (A_{r+1}). Moreover, $\boldsymbol{g}_r$ being obtained by composing the isomorphisms noted $\boldsymbol{v}$ and $\boldsymbol{u}$ above, we have $g_{ri}(\boldsymbol{x}) = x_i + h_{ri}(\boldsymbol{x})$ $(1 \leqq i \leqq n)$, where in the power series h_{ri}, all monomials have a height $\geqq r+1$. As usual, we can now therefore consider the "infinite product" $\cdots \boldsymbol{g}_r \boldsymbol{g}_{r-1} \cdots \boldsymbol{g}_0$, with $G_0 = G$; this is an isomorphism $\boldsymbol{g}$ of G onto a group G_∞ such that the group law of G_∞ is pseudo-canonical, and $\mathfrak{H}_\infty = \boldsymbol{g}'(\mathfrak{H})$ now satisfies the assumptions of Theorem 1, and assumptions (A_r) for *every* r. This shows that we may suppose that, by a suitable isomorphism, the Z_α such that $\alpha \in \boldsymbol{N}^J$ form a basis for $\mathfrak{H}$; but then property 1° shows that these Z_α constitute the structural basis of a hyperalgebra [10, p. 59, prop. 1] and this ends the proof of Theorem 1.

We may observe here that Theorem 1 obviously implies Theorem 1 of [13], of which we thus obtain an alternate proof.

A similar method enables one to prove

PROPOSITION 4. *Let $H_1 \subset H_2 \subset \cdots \subset H_s$ be a strictly increasing sequence of subgroups*[5] *of G, of dimensions $m_1 < m_2 < \cdots < m_s \leqq n$; let J_k be the interval $1 \leqq i \leqq m_k$ $(1 \leqq k \leqq s)$. By applying a suitable isomorphism to G, it is possible to suppose that the group law of G is pseudo-canonical and that, for each k $(1 \leqq k \leqq s)$, the subgroup H_k is typical and defined by the equations $x_i = 0$ for $i \notin J_k$.*

We only sketch the proof. First, by a linear isomorphism, one can suppose that for each k $(1 \leqq k \leqq s)$, the X_{0i} such that $i \in J_k$ form a basis for $\mathfrak{g}_0 \cap \mathfrak{H}_k$. Then one builds up an inductive proof by supposing that the group law of G is pseudo-canonical and that for each k $(1 \leqq k \leqq s)$ the elements X_α such that $\alpha \in \boldsymbol{N}^{J_k}$ and $h(\alpha) \leqq r$ form a basis for $\mathfrak{s}_r \cap \mathfrak{H}_k$. One then defines the $Y_{r+1,i}$ $(1 \leqq i \leqq n)$ such that $Y_{r+1,i} - X_{r+1,i} \in \mathfrak{g}_0$, first for $i \in J_1$ by the

[5] From now on, we will most of the time omit the mention of the monomorphism f in the notation of a subgroup (H, f), and simply write it H; we will refer to f as the "natural injection" of H into G. Similarly, when we will speak of the "restriction to H" of a homomorphism u of G into a group G', we mean the homomorphism $u \circ f$ of H into G'; etc.

condition that $Y_{r+1,i} \in \mathfrak{g}_{r+1} \cap \mathfrak{H}_1$, then for $i \in J_2 - J_1$ by the condition $Y_{r+1,i} \in \mathfrak{g}_{r+1} \cap \mathfrak{H}_2$ etc. We omit the remainder of the proof, which follows very closely that of Theorem 1.

We define the *dimension* of a subhyperalgebra $\mathfrak{H}$ of $\mathfrak{G}$ as the dimension of the subgroup of G of which $\mathfrak{H}$ is the subhyperalgebra (no confusion can arise with the dimension of $\mathfrak{H}$ as a vector space, which is always infinite); it is equal to the dimension of the Lie algebra $\mathfrak{g}_0 \cap \mathfrak{H}$ over K. If $\mathfrak{H}$, $\mathfrak{H}'$ are two subhyperalgebras of $\mathfrak{G}$, such that $\mathfrak{H} \subset \mathfrak{H}'$, then $\dim \mathfrak{H} \leqq \dim \mathfrak{H}'$, and the relation $\dim \mathfrak{H} = \dim \mathfrak{H}'$ implies $\mathfrak{H} = \mathfrak{H}'$, as follows from Proposition 3. An obvious consequence of these remarks is that the subhyperalgebras of $\mathfrak{G}$ verify *both chain conditions* for the inclusion relation.

It is natural to ask if the subgroups of a formal Lie group (or the corresponding subhyperalgebras) form a *lattice* for the inclusion relation. We will be able to show that such is the case, but only *via* a very indirect argument (no. 17). The difficulties which block a direct approach are the following ones: it is very easy to give examples (for a product G of three additive groups) of subhyperalgebras $\mathfrak{H}_1$, $\mathfrak{H}_2$ of $\mathfrak{G}$ such that $\mathfrak{H}_1 \cap \mathfrak{H}_2$ *is not a subhyperalgebra*; one may ask if the *subalgebra* of $\mathfrak{G}$ generated by $\mathfrak{H}_1 \cup \mathfrak{H}_2$ is a subhyperalgebra; I will prove in no. 9 that such is the case when one of the subgroups H_1, H_2 is normal, but the proof will clearly show the obstacles to the extension of that result (if true) to the general case.

As soon as we have proved that the l. u. b. (resp. g. l. b.) of two subgroups H_1, H_2 exists, we will write it $H_1 \vee H_2$ (resp. $H_1 \wedge H_2$), and use similar notations for the l. u. b. and g. l. b. of subhyperalgebras.

II. Normal Subgroups and Quotient Groups.

6. Inner automorphisms and normal subgroups. Let G be a formal Lie group of dimension n over K; we know [8, p. 88] that there exists a unique system θ of n formal power series $\theta_i(\boldsymbol{x})$ $(1 \leqq i \leqq n)$ such that $\phi(\boldsymbol{x}, \theta(\boldsymbol{x})) = \phi(\theta(\boldsymbol{x}), \boldsymbol{x}) = \boldsymbol{e}$. If we write $\boldsymbol{x}^{-1}$ for that system of n power series,[6] we can therefore define, for two independent generic points $\boldsymbol{x}, \boldsymbol{z}$ of G, $\boldsymbol{zxz}^{-1}$ as the system of n power series $\phi_i(\phi(\boldsymbol{z}, \boldsymbol{x}), \theta(\boldsymbol{z}))$; if $\boldsymbol{y}$ is another independent generic point of G, we have $(\boldsymbol{zxz}^{-1})(\boldsymbol{zyz}^{-1}) = \boldsymbol{z}(\boldsymbol{xy})\boldsymbol{z}^{-1}$ by the usual associativity argument. Let L be a perfect field containing the ring $K[[\boldsymbol{z}]] = K[[z_1, \cdots, z_n]]$ of power series in the z_i with coefficients in K;

[6] The relations $(\boldsymbol{x}^{-1})^{-1} = \boldsymbol{x}$, $(\boldsymbol{xy})^{-1} = \boldsymbol{y}^{-1}\boldsymbol{x}^{-1}$, $\boldsymbol{u}(\boldsymbol{x}^{-1}) = (\boldsymbol{u}(\boldsymbol{x}))^{-1}$ (for a homomorphism $\boldsymbol{u}$) are proved by the standard arguments.

we can say that the system $\boldsymbol{i}_z(\boldsymbol{x}) = \boldsymbol{z}\boldsymbol{x}\boldsymbol{z}^{-1}$ of n power series in $x_1, \cdots, x_n$, with coefficients in L, defines an *automorphism* of the group $G_{(L)}$, which we will call an *inner automorphism* of $G_{(L)}$ (or of G by abuse of language); its derivative $\boldsymbol{i}'_z$ is an automorphism of the hyperalgebra $\mathfrak{G}_{(L)}$.

More generally, if $\boldsymbol{u}$ is a homomorphism of a group H into G, $\boldsymbol{y}$ a generic point of H and L a perfect field containing $K[[\boldsymbol{y}]]$, the system $\boldsymbol{i}_{\boldsymbol{u}(\boldsymbol{y})}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{y})\boldsymbol{x}(\boldsymbol{u}(\boldsymbol{y}))^{-1}$ of n power series in the x_i, with coefficients in L, defines an automorphism of $G_{(L)}$.

Let $(H, \boldsymbol{f})$ be a subgroup of G, $\mathfrak{H}$ the corresponding subhyperalgebra of $\mathfrak{G}$. We say that $(H, \boldsymbol{f})$ is a *normal subgroup* of G if the subgroup $(H_{(L)}, \boldsymbol{i}_z \circ \boldsymbol{f})$ of $G_{(L)}$ is contained in $(H_{(L)}, \boldsymbol{f})$, in the sense of no. 5; by Proposition 3, an equivalent condition is that $\boldsymbol{i}'_z(\mathfrak{H}) \subset \mathfrak{H}_{(L)}$. If we suppose that $(H, \boldsymbol{f})$ is a *typical* subgroup defined by $x_{m+1} = \cdots = x_n = 0$, then in order that such a subgroup be normal, necessary and sufficient conditions are that in the system $\boldsymbol{i}_z(\boldsymbol{x}) = (h_1(\boldsymbol{x}), \cdots, h_n(\boldsymbol{x}))$ of n power series in the x_i and z_i, the n-m last series be 0.

It is clear that $\boldsymbol{i}'_z(\mathfrak{H}_{(L)}) \subset \mathfrak{H}_{(L)}$, and as these two subhyperalgebras have same dimension over L, we have $\boldsymbol{i}'_z(\mathfrak{H}_{(L)}) = \mathfrak{H}_{(L)}$. We also note that if H is a normal subgroup of G, then, for any perfect extension E of K, $H_{(E)}$ is a normal subgroup of $G_{(E)}$.

7. Kernels and normal subgroups. Let $\boldsymbol{u}$ be a homomorphism of rank ρ of a group G of dimension n into a group G'; if $\boldsymbol{u}$ has the form (2), we have defined in [8, p. 114], the *kernel* of $\boldsymbol{u}$ as the typical subgroup of G, of dimension $n-\rho$, defined by the relations $x_i = 0$ for $i > \rho$. We will first see that this notion is an intrinsic one:

PROPOSITION 5. *The subhyperalgebra $\mathfrak{N}$ of $\mathfrak{G}$ corresponding to the kernel of $\boldsymbol{u}$ is the largest subhyperalgebra $\mathfrak{H}$ such that $\boldsymbol{u}'(\mathfrak{H}) = \{0\}$.*

As $\boldsymbol{u}'(\mathfrak{N}) = \{0\}$, it is equivalent to prove that if $\boldsymbol{u}'(\mathfrak{H}) = \{0\}$, then $\mathfrak{H} \subset \mathfrak{N}$. Let us observe that, from (3), it follows that each X'_{ki} is of the form $\boldsymbol{u}'(X_{ki})$ for a *unique* index k; let us order the factors X'_{ki} in the X'_λ in the same order as the corresponding factors X_{ki} have in the products X_α; then those of the $\boldsymbol{u}'(X_\alpha)$ which are not 0 are *distinct* elements X'_λ, hence linearly independent. We prove first that $\mathfrak{H} \cap \mathfrak{g}_0 \subset \mathfrak{N} \cap \mathfrak{g}_0$; indeed, let us suppose there is in $\mathfrak{H} \cap \mathfrak{g}_0$ an element $\sum_{i=1}^{n} c_i X_{0i}$ with at least one of the c_i of index $i \leqq \rho$ being $\neq 0$. As $\mathfrak{H}$ is a subhyperalgebra, condition 3° of Theorem 1 shows that there exists in $\mathfrak{H}$ an element of the form $Y = \sum_{i=1}^{n} c^{p^t}_i X_{ti} + \sum_\lambda d_\lambda X_\lambda$

the summation being extended over indices of height $< t$ and such that $\boldsymbol{p}'(X_\lambda) = 0$. If we now observe that by (3), none of the $\boldsymbol{u}'(X_{ti})$ with $i \leqq \rho$ is 0, it follows from the preceding remark that we cannot have $\boldsymbol{u}'(Y) = 0$, contradicting the hypothesis. From that result and conditions 2° of Theorem 1 it follows that $\mathfrak{H} \cap \mathfrak{s}_0 \subset \mathfrak{N} \cap \mathfrak{s}_0$. Suppose now that we have proved that $\mathfrak{s}_r \cap \mathfrak{H} \subset \mathfrak{s}_r \cap \mathfrak{N}$. From the argument in the beginning of the proof of Theorem 1 (based on condition 1°), we see that any element of $\mathfrak{g}_{r+1} \cap \mathfrak{H}$ can be written $Z + \sum_{i=1}^{n} c_i X_{0i}$, with $Z \in \mathfrak{N} \cap \mathfrak{g}_{r+1}$; then $\mathfrak{H}$ also contains an element $Y = \sum_{i=1}^{n} c^{p^t} {}_i X_{ti} + T + \sum_\lambda d_\lambda X_\lambda$, with $T \in \mathfrak{N}$ and $\boldsymbol{p}'^t(X_\lambda) = 0$ for all indices λ in the summation. If one of the c_i of index $\leqq \rho$ was not 0, we would then get $\boldsymbol{u}'(Y) \neq 0$ as above, contradicting again the hypothesis. Therefore $\mathfrak{g}_{r+1} \cap \mathfrak{H} \subset \mathfrak{g}_{r+1} \cap \mathfrak{N}$, hence $\mathfrak{s}_{r+1} \cap \mathfrak{H} \subset \mathfrak{s}_{r+1} \cap \mathfrak{N}$ by condition 2° of Theorem 1, and Proposition 5 is thus proved by induction on r.

THEOREM 2. *Let $(N, \boldsymbol{f})$ be a subgroup of G, $\mathfrak{N}$ the corresponding sub-hyperalgebra of $\mathfrak{G}$. The following properties are equivalent:*

a) *$(N, \boldsymbol{f})$ is a normal subgroup;*

b) *the left (or right) ideal generated in $\mathfrak{G}$ by $\mathfrak{N}^+$ is a two-sided ideal;*

c) *$(N, \boldsymbol{f})$ is the kernel of an epimorphism;*

d) *$(N, \boldsymbol{f})$ is the kernel of a homomorphism.*

It is clear that c) implies d); condition d) implies a), for if $\boldsymbol{u}(\boldsymbol{f}(\boldsymbol{y})) = \boldsymbol{e}'$ for a homomorphism $\boldsymbol{u}$ of G into G' ($\boldsymbol{y}$ generic point of N), we also have, for a generic point $\boldsymbol{z}$ of G independent of $\boldsymbol{y}$, $\boldsymbol{u}(\boldsymbol{z}\boldsymbol{f}(\boldsymbol{y})\boldsymbol{z}^{-1}) = \boldsymbol{u}(\boldsymbol{z})\boldsymbol{u}(\boldsymbol{f}(\boldsymbol{y}))(\boldsymbol{u}(\boldsymbol{z}))^{-1} = \boldsymbol{e}'$, hence by Proposition 5, the subgroup $(N_{(L)}, \boldsymbol{i}_z \circ \boldsymbol{f})$ is contained in $(N_{(L)}, \boldsymbol{f})$, which proves that $(N, \boldsymbol{f})$ is normal.

To prove that a) implies b), we can suppose that the group law of G is pseudo-canonical and that N is a typical subgroup defined by the relations $x_1 = \cdots = x_m = 0$; we will denote by I the interval $1 \leqq i \leqq m$, by J the interval $m+1 \leqq i \leqq n$. Let $\boldsymbol{y} = (y_{m+1}, \cdots, y_n)$; then $\boldsymbol{f}(\boldsymbol{y}) = (0, \cdots, 0, y_{m+1}, \cdots, y_n)$, and if, for simplicity's sake, we write $\boldsymbol{z}^{-1}\boldsymbol{f}(\boldsymbol{y})\boldsymbol{z} = \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z})$, we have by assumption $\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}) = (0, \cdots, 0, h_{m+1}(\boldsymbol{y}, \boldsymbol{z}), \cdots, h_n(\boldsymbol{y}, \boldsymbol{z}))$. By definition, $\boldsymbol{f}(\boldsymbol{y})\boldsymbol{z} = \boldsymbol{z}\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z})$; remembering that $(\phi(\boldsymbol{x}, \boldsymbol{z}))^\gamma = \sum_{\alpha,\beta} d_{\alpha\beta\gamma} \boldsymbol{x}^\alpha \boldsymbol{z}^\beta$ for two independent generic points $\boldsymbol{x}, \boldsymbol{z}$ of G, we obtain, for any index γ

$$(8) \qquad \sum_{\alpha,\beta} d_{\alpha\beta\gamma} (\boldsymbol{f}(\boldsymbol{y}))^\alpha \boldsymbol{z}^\beta = \sum_{\lambda,\mu} d_{\lambda\mu\gamma} \boldsymbol{z}^\lambda (\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}))^\mu.$$

Now $(\boldsymbol{f}(\boldsymbol{y}))^\alpha = 0$ unless $\alpha \in \boldsymbol{N}^J$, and similarly $(\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}))^\mu = 0$ unless

$\mu \in \boldsymbol{N}^J$. Let α be an arbitrary index in $\boldsymbol{N}^J$, and β an arbitrary index in $\boldsymbol{N}^I$, and let us identify in (8) the unique monomial $d_{\alpha\beta\gamma}(\boldsymbol{f}(\boldsymbol{y}))^{\alpha}\boldsymbol{z}^{\beta}$ to the sum of the monomials of same degrees in the y_i and z_j on the right hand side. It is clear that such terms can only come from products $\boldsymbol{z}^{\lambda}(\boldsymbol{h}(\boldsymbol{y},\boldsymbol{z}))^{\mu}$ such that $\lambda \in \boldsymbol{N}^I$; hence, we can write

$$d_{\alpha\beta\gamma} = \sum_{\lambda,\mu} a_{\lambda\mu} d_{\lambda\mu\gamma} \tag{9}$$

where on the right hand side, the summation extends only over pairs (λ, μ) *such that* $\lambda \in \boldsymbol{N}^I$ *and* $\mu \in \boldsymbol{N}^J$. But from Lemma 1 it follows that under those conditions $Z_\lambda Z_\mu = Z_{\lambda+\mu}$, hence $d_{\lambda\mu\gamma} = 0$ except if $\gamma = \lambda + \mu$, and such a decomposition of an index γ in a sum of $\lambda \in \boldsymbol{N}^I$ and $\mu \in \boldsymbol{N}^J$ is unique, hence $d_{\alpha\beta\gamma} = a_{\lambda\mu} d_{\lambda\mu\gamma}$. On the other hand, if $\alpha \neq 0$ in $\boldsymbol{N}^J$, then we must have $\mu \neq 0$, otherwise the term $\boldsymbol{z}^{\lambda}$ cannot contain any y_j. We conclude that if $\alpha \in \boldsymbol{N}^J$, $\beta \in \boldsymbol{N}^I$ and $\alpha \neq 0$, then $d_{\alpha\beta\gamma} = 0$ *unless* $\gamma = \lambda + \mu$ *with* $\lambda \in \boldsymbol{N}^I$, $\mu \in \boldsymbol{N}^J$ *and* $\mu \neq 0$.

Now from Lemma 1 it is clear that the left ideal of $\mathfrak{G}$ generated by the subalgebra $\mathfrak{N}^+$ has as a basis over K the elements $Z_{\lambda+\mu}$ with $\lambda \in \boldsymbol{N}^I$, $\mu \in \boldsymbol{N}^J$ and $\mu \neq 0$. As $Z_\alpha Z_\beta = \sum_\gamma d_{\alpha\beta\gamma} Z_\gamma$, the preceding result shows that the right ideal generated by the subalgebra $\mathfrak{N}^+$ is contained in the left ideal generated by that subalgebra, in other words the latter is a two-sided ideal. A similar argument, applied to the group opposite to G, proves that the right ideal generated by $\mathfrak{N}^+$ is the same two-sided ideal as the left ideal generated by $\mathfrak{N}^+$.

Finally, to prove that b) implies c), consider (under the same assumptions regarding the group law of G and the typical form of N) the two-sided ideal $\mathfrak{a}$ generated by $\mathfrak{N}^+$, which has, as we have seen, the $Z_{\lambda+\mu}$ with $\lambda \in \boldsymbol{N}^I$, $\mu \in \boldsymbol{N}^J$ and $\mu \neq 0$, as a basis over K; the elements Z_λ such that $\lambda \in \boldsymbol{N}^I$ span therefore a supplementary subspace to $\mathfrak{a}$ in $\mathfrak{G}$. If $\boldsymbol{s}$ is the natural mapping of $\mathfrak{G}$ onto $\mathfrak{G}/\mathfrak{a} = \mathfrak{G}'$, the elements $\bar{Z}_\lambda = \boldsymbol{s}(Z_\lambda)$ with $\lambda \in \boldsymbol{N}^I$ thus constitute a basis for $\mathfrak{G}'$; furthermore, we have

$$(\boldsymbol{s} \otimes \boldsymbol{s})(Z^0{}_\lambda) = \sum_{0 \leqq \nu \leqq \lambda} \boldsymbol{s}(Z_\nu) \otimes \boldsymbol{s}(Z_{\lambda-\nu})$$

hence $(\boldsymbol{s} \otimes \boldsymbol{s})(Z^0{}_\lambda) = 0$ unless $\lambda \in \boldsymbol{N}^I$. This proves that the image of $\mathfrak{G}'$ into $\mathfrak{G}' \otimes \mathfrak{G}'$ by the linear mapping which to each $\bar{Z}_\lambda$ associates $\bar{Z}^0{}_\lambda = \sum_{0 \leqq \nu \leqq \lambda} \bar{Z}_\nu \otimes \bar{Z}_{\lambda-\nu}$ ($\lambda \in \boldsymbol{N}^I$) is the subalgebra $\mathfrak{G}'^0 = (\boldsymbol{s} \otimes \boldsymbol{s})(\mathfrak{G}^0)$. Moreover, it is clear that the $\bar{Z}_\lambda$ with $\lambda \neq 0$ constitute a basis for a two-sided ideal of $\mathfrak{G}'$, and the $\bar{Z}_\lambda$ of height $h(\lambda) \leqq r$ a basis of a subalgebra of $\mathfrak{G}'$. Applying Proposition 1 of [10, p. 59], we see that $(\bar{Z}_\lambda)_{\lambda \in \boldsymbol{N}^I}$ is a structural basis for a well determined

structure of hyperalgebra on $\mathfrak{G}'$. We will say that the corresponding formal Lie group of dimension $n-m$ is the *quotient group* of G by N, and write it G/N [7]; $\boldsymbol{s}$ is then the derived mapping of an *epimorphism* $\boldsymbol{g}$ of G onto G/N, having kernel N, and which we will call the *natural mapping* of G onto G/N. This concludes the proof of Theorem 2.

We will say that the two-sided ideal $\mathfrak{a}$ generated by $\mathfrak{N}^+$ (which is also the right and the left ideal generated by $\mathfrak{N}^+$) is the *normal ideal* corresponding to N.

The following results are immediate consequences of Theorem 2:

COROLLARY 1. *If N is a normal subgroup of G, $\mathfrak{g}_0 \cap \mathfrak{N}$ is a p-ideal in $\mathfrak{g}_0$, and $\mathfrak{g}_0/(\mathfrak{g}_0 \cap \mathfrak{N})$ is isomorphic to the Lie algebra of G/N.*

COROLLARY 2. *If N is a normal subgroup of G, then, for any perfect extension L of K, $(G/N)_{(L)}$ is identified to $G_{(L)}/N_{(L)}$.*

COROLLARY 3. *If L is a perfect extension of K, and $\mathfrak{a}$ a normal ideal corresponding to a normal subgroup N of G, then the vector space $\mathfrak{a}_{(L)}$ generated by $\mathfrak{a}$ in $\mathfrak{G}_{(L)}$ is the normal ideal corresponding to the normal subgroup $N_{(L)}$ of $G_{(L)}$.*

Let now $\boldsymbol{u}$ be a homomorphism of G into a group G'', having N as its kernel, and reduced to the form (2); we have $\boldsymbol{u}'(Z_\lambda) = 0$ except for $\lambda \in \boldsymbol{N}^I$ and therefore, for λ and μ in $\boldsymbol{N}^I$,

$$\boldsymbol{u}'(Z_\lambda)\boldsymbol{u}'(Z_\mu) = \sum_\nu d_{\lambda\mu\nu}\boldsymbol{u}'(Z_\nu) \qquad (\nu \in \boldsymbol{N}^I). \tag{10}$$

On the other hand, the definition of the $\bar{Z}_\lambda$ yields

$$\bar{Z}_\lambda \bar{Z}_\mu = \sum_\nu d_{\lambda\mu\nu}\bar{Z}_\nu \qquad (\nu \in \boldsymbol{N}^I). \tag{11}$$

Observe that, from the homomorphism theorem, it follows that $\boldsymbol{u}'(\mathfrak{G}) = \mathfrak{L}$ is a subhyperalgebra of $\mathfrak{G}''$. Let $\boldsymbol{t}$ be the linear mapping of $\mathfrak{G}'$ into $\mathfrak{L}$ defined by $\boldsymbol{t}(\bar{Z}_\lambda) = \boldsymbol{u}'(Z_\lambda)$ $(\lambda \in \boldsymbol{N}^I)$; then we have, by the preceding definitions

$$(\boldsymbol{t} \otimes \boldsymbol{t})(\bar{Z}^0{}_\lambda) = \sum_{0 \leqq \nu \leqq \lambda} \boldsymbol{u}'(Z_\nu) \otimes \boldsymbol{u}'(Z_{\lambda-\nu}) = (\boldsymbol{u}' \otimes \boldsymbol{u}')(Z^0{}_\lambda) \in \mathfrak{L} \otimes \mathfrak{L}$$

and on the other hand it follows from (10) and (11) that $\boldsymbol{t}$ is a homomorphism of $\mathfrak{G}'$ into $\mathfrak{L}$; therefore [10, p. 59, prop. 2] $\boldsymbol{t}$ is the derivative of

[7] Strictly speaking, this group is thus only defined when the group law of G is pseudo-canonical and N is a typical normal subgroup. By abuse of language, we will designate by G/N any formal Lie group isomorphic to G'/N', where G' is a group with pseudo-canonical group law, isomorphic to G by an isomorphism which transforms N into a normal typical subgroup N'.

a homomorphism $\boldsymbol{v}$ of G/N onto a group L of hyperalgebra $\mathfrak{L}$. Moreover, there is a monomorphism $\boldsymbol{f}$ of L into G'' and G/N and L have the same dimension, hence $\boldsymbol{v}$ is an isogeny; in other words:

PROPOSITION 6. *Any homomorphism $\boldsymbol{u}$ of G into G'', of kernel N, can be decomposed as $\boldsymbol{f}\circ\boldsymbol{v}\circ\boldsymbol{g}$, where $\boldsymbol{g}$ is the natural epimorphism of G onto G/N, $\boldsymbol{v}$ an isogeny of G/N onto a group L, and $\boldsymbol{f}$ a monomorphism of L into G''.*

8. Direct and inverse images of subgroups. We have just observed that if $\boldsymbol{u}$ is a homomorphism of G into G', $\boldsymbol{u}'(\mathfrak{G})$ is a subhyperalgebra of $\mathfrak{G}'$. More generally, it follows that if $(H,\boldsymbol{f})$ is a subgroup of G, $\boldsymbol{u}'(\boldsymbol{f}'(\mathfrak{H}))$ is a subhyperalgebra of $\mathfrak{G}'$, since $\boldsymbol{u}\circ\boldsymbol{f}$ is a homomorphism of H into G'; we will say that the corresponding subgroup of G' is the *direct image* of the subgroup $(H,\boldsymbol{f})$ of G by $\boldsymbol{u}$, and we will write it $\boldsymbol{u}(H)$ by abuse of language when we drop the monomorphism $\boldsymbol{f}$ in the notation of a subgroup of G.

PROPOSITION 7. *If $\boldsymbol{u}$ is an isogeny of a group G onto a group G', the mapping $H\to\boldsymbol{u}(H)$ is a bijection of the set of subgroups* (resp. *normal subgroups*) *of G onto the set of subgroups* (resp. *normal subgroups*) *of G'.*

Consider first the case in which $G'=G^{(t)}$ and $\boldsymbol{u}=\boldsymbol{p}^t$; then it is clear that $\boldsymbol{u}(H)=H^{(t)}$ for any subgroup H of G, as it is immediately seen when H is taken as typical subgroup; Proposition 7 is just obvious in that case. If now $\boldsymbol{u}$ is any isogeny, there is an isogeny $\boldsymbol{v}$ of G' onto some $G^{(t)}$ such that $\boldsymbol{v}\circ\boldsymbol{u}=\boldsymbol{p}^t$ (Proposition 2), and as $\boldsymbol{u}^{(t)}\circ\boldsymbol{v}=\boldsymbol{p}^t$, the result follows at once.

PROPOSITION 8. *Let $\boldsymbol{u}$ be a surjective homomorphism of G onto G'. For any subgroup H' of G', there is a largest subgroup H of G such that $\boldsymbol{u}(H)\subset H'$; H is the unique subgroup of G containing the kernel N of $\boldsymbol{u}$ and such that $\boldsymbol{u}(H)=H'$, and we have* $\dim H=\dim N+\dim H'$.

It is clear that if there is a largest subgroup H of G such that $\boldsymbol{u}(H)\subset H'$, it must contain N; for any subgroup F of G such that $N\subset F\subset H$, N is then the kernel of the restriction of $\boldsymbol{u}$ to F, and we have, by the homomorphism theorem, $\dim F=\dim N+\dim\boldsymbol{u}(F)$; the proposition will therefore be proved as soon as we establish the existence of a subgroup $H\supset N$ of G such that $\boldsymbol{u}(H)=H'$, and the fact that H is the largest of the subgroups F for which $\boldsymbol{u}(F)\subset H'$.

By Proposition 6, we can write $\boldsymbol{u}=\boldsymbol{v}\circ\boldsymbol{w}$, where $\boldsymbol{w}$ is an epimorphism and $\boldsymbol{v}$ an isogeny; using Proposition 7, we are immediately reduced to the case in which $\boldsymbol{u}$ is an *epimorphism*. Suppose $\dim G=n$, $\dim G'=m\leqq n$,

and (using the homomorphism theorem) suppose $\boldsymbol{u}$ is defined by $u_i(\boldsymbol{x}) = x_i$ for $1 \leqq i \leqq m$, $u_i(\boldsymbol{x}) = 0$ for $m+1 \leqq i \leqq n$. We can change variables in G', by an isomorphism $\bar{x}'_i = f_i(x'_1, \cdots, x'_m)$ $(1 \leqq i \leqq m)$ so that H' will be transformed into a typical subgroup defined by $\bar{x}'_i = 0$ for $1 \leqq i \leqq q \leqq m$. If we also change variables in G by $\bar{x}_i = f_i(x_1, \cdots, x_m)$ for $1 \leqq i \leqq m$, $\bar{x}_i = x_i$ for $m+1 \leqq i \leqq n$, then the epimorphism $\boldsymbol{u}$ corresponding to $\boldsymbol{u}$ will again be defined by $\bar{u}_i(\boldsymbol{x}) = \bar{x}_i$ for $1 \leqq i \leqq m$, $\bar{u}_i(\boldsymbol{x}) = 0$ for $m+1 \leqq i \leqq n$. We can therefore suppose that H' was already a typical subgroup of G' defined by $x'_i = 0$ for $1 \leqq i \leqq q$. I claim that the relations $x_i = 0$ for $1 \leqq i \leqq q$ define a typical subgroup H of G, which then obviously contains N and is such that $\boldsymbol{u}(H) = H'$. We have to show that

$$\phi_j(0, \cdots, 0, x_{q+1}, \cdots, x_n, 0, \cdots, 0, y_{q+1}, \cdots, y_n) = 0 \text{ for } j \leqq q.$$

But by assumption we have

$$\phi'_j(0, \cdots, 0, x'_{q+1}, \cdots, x'_m, 0, \cdots, 0, y'_{q+1}, \cdots, y'_m) = 0 \text{ for } j \leqq q.$$

Now, by definition of a homomorphism, we have

$$\phi'_j(x_1, \cdots, x_m, y_1, \cdots, y_m) = \phi_j(x_1, \cdots, x_n, y_1, \cdots, y_n) \text{ for } 1 \leqq j \leqq m;$$

substituting 0 to the x_k and y_k of index $k \leqq q$ in those identities for $1 \leqq j \leqq q$ yields the desired result.

It remains to prove that if a subhyperalgebra $\mathfrak{F}$ of $\mathfrak{G}$ is such that $\boldsymbol{u}'(\mathfrak{F}) \subset \mathfrak{H}'$, then $\mathfrak{F} \subset \mathfrak{H}$. First we have $\boldsymbol{u}'(\mathfrak{F} \cap \mathfrak{g}_0) \subset \mathfrak{H}' \cap \mathfrak{g}'_0$; as $\boldsymbol{u}'(X_{0i}) = X'_{0i}$ for $1 \leqq i \leqq m$, $\boldsymbol{u}'(X_{0i}) = 0$ for $m+1 \leqq i \leqq n$, this shows that $\mathfrak{F} \cap \mathfrak{g}_0 \subset \mathfrak{H} \cap \mathfrak{g}_0$, hence $\mathfrak{F} \cap \mathfrak{s}_0 \subset \mathfrak{H} \cap \mathfrak{s}_0$. Suppose we have proved that $\mathfrak{F} \cap \mathfrak{s}_r \subset \mathfrak{H} \cap \mathfrak{s}_r$; by the argument at the beginning of the proof of Theorem 1, any element of $\mathfrak{F} \cap \mathfrak{g}_{r+1}$ can then be written $Z + \sum_{i=1}^{n} c_i X_{0i}$, with $Z \in \mathfrak{H} \cap \mathfrak{g}_{r+1}$; as $\boldsymbol{u}'(Z) \in \mathfrak{H}'$, we must have $c_i = 0$ for $1 \leqq i \leqq q$, hence $\sum_{i=1}^{n} c_i X_{0i} \in \mathfrak{H}$, which proves that $\mathfrak{F} \cap \mathfrak{g}_{r+1} \subset \mathfrak{H} \cap \mathfrak{g}_{r+1}$, hence $\mathfrak{F} \cap \mathfrak{s}_{r+1} \subset \mathfrak{H} \cap \mathfrak{s}_{r+1}$ (by condition 2° of Theorem 1), which ends the proof.

We will write as usual $H = \boldsymbol{u}^{-1}(H')$, and say that H is the *inverse image of* H' by $\boldsymbol{u}$.

COROLLARY 1. *If $\boldsymbol{u}$ is a surjective homomorphism of G onto G', $\boldsymbol{v}$ a surjective homomorphism of G' onto G'' and $\boldsymbol{w} = \boldsymbol{v} \circ \boldsymbol{u}$, then, for any subgroup H'' of G'', $\boldsymbol{w}^{-1}(H'') = \boldsymbol{u}^{-1}(\boldsymbol{v}^{-1}(H''))$.*

For if $H = \boldsymbol{u}^{-1}(\boldsymbol{v}^{-1}(H''))$ we have $\boldsymbol{w}(H) = H''$ by Proposition 8; on the other hand if N' is the kernel of $\boldsymbol{v}$, $\boldsymbol{u}^{-1}(N')$ is the kernel of $\boldsymbol{w}$, for the

relation $\boldsymbol{v}(\boldsymbol{u}(F)) = \{e''\}$ implies $\boldsymbol{u}(F) \subset N'$ by Proposition 5 for a subgroup F of G, hence $F \subset \boldsymbol{u}^{-1}(N')$, and conversely $\boldsymbol{v}(\boldsymbol{u}(\boldsymbol{u}^{-1}(N'))) = \boldsymbol{v}(N') = \{e''\}$. As $N' \subset \boldsymbol{v}^{-1}(H'')$, we have $\boldsymbol{u}^{-1}(N') \subset \boldsymbol{u}^{-1}(\boldsymbol{v}^{-1}(H''))$, and Proposition 8 shows that $H = \boldsymbol{w}^{-1}(H'')$.

COROLLARY 2. *If L is a perfect extension of K, $\boldsymbol{u}^{-1}(H'_{(L)})$ is identical to $(\boldsymbol{u}^{-1}(H'))_{(L)}$.*

For if $H = \boldsymbol{u}^{-1}(H')$, $\boldsymbol{u}(H_{(L)}) = H'_{(L)}$, and $H_{(L)}$ contains the kernel $N_{(L)}$ of $\boldsymbol{u}$ (considered as homomorphism of $G_{(L)}$ onto $G'_{(L)}$).

If the homomorphism $\boldsymbol{u}$ is not surjective, we can still prove the existence of a largest subgroup H of G such that $\boldsymbol{u}(H) \subset H'$ as soon as we have proved the g.l.b. $H' \wedge \boldsymbol{u}(G)$ exists; for the relation $\boldsymbol{u}(H) \subset H'$ is then equivalent to $\boldsymbol{u}(H) \subset H' \wedge \boldsymbol{u}(G)$, and considering $\boldsymbol{u}$ as a surjective homomorphism of G onto $\boldsymbol{u}(G)$, we need only take $H = \boldsymbol{u}^{-1}(H' \wedge \boldsymbol{u}(G))$

9. Lattice properties of normal subgroups.

PROPOSITION 9. *Let H be a subgroup of G, N a normal subgroup of G. Then there is a* g.l.b. $H \wedge N$ *and a* l.u.b. $H \vee N$ *of these two subgroups. If $\boldsymbol{g}$ is the natural epimorphism of G onto G/N, $H \wedge N$ is the kernel of the restriction of $\boldsymbol{g}$ to H, and $H \vee N = \boldsymbol{g}^{-1}(\boldsymbol{g}(H))$.*

The fact that the kernel of the restriction of $\boldsymbol{g}$ to H is $H \wedge N$ follows from Proposition 5. On the other hand, if a subgroup F of G contains both H and N, we have $\boldsymbol{g}(F) \supset \boldsymbol{g}(H)$, hence $\boldsymbol{g}^{-1}(\boldsymbol{g}(F)) \supset \boldsymbol{g}^{-1}(\boldsymbol{g}(H))$. But by Proposition 8 $F = \boldsymbol{g}^{-1}(\boldsymbol{g}(F))$, which proves that $\boldsymbol{g}^{-1}(\boldsymbol{g}(H)) = H \vee N$.

COROLLARY 1. *For any perfect extension L of K,*

$$H_{(L)} \wedge N_{(L)} = (H \wedge N)_{(L)} \text{ and } H_{(L)} \vee N_{(L)} = (H \vee N)_{(L)}.$$

It is clear that $(H \wedge N)_{(L)}$ is the kernel of the restriction of $\boldsymbol{g}$ to $H_{(L)}$; on the other hand, from corollary 2 to Proposition 8 we have $(\boldsymbol{g}^{-1}(\boldsymbol{g}(H)))_{(L)} = \boldsymbol{g}^{-1}((\boldsymbol{g}(H))_{(L)})$, and consideration of the subhyperalgebras shows that $(\boldsymbol{g}(H))_{(L)} = \boldsymbol{g}(H_{(L)})$.

COROLLARY 2. *If N_1, N_2 are two normal subgroups of G, $N_1 \wedge N_2$ and $N_1 \vee N_2$ are normal in G.*

Let $\boldsymbol{z}$ be a generic point of G, and L a perfect field containing $K[[z_1, \cdots, z_n]]$; then $\boldsymbol{i}_{\boldsymbol{z}}((N_1 \wedge N_2)_{(L)}) \subset \boldsymbol{i}_{\boldsymbol{z}}(N_{1(L)}) = N_{1(L)}$ and similarly $\boldsymbol{i}_{\boldsymbol{z}}((N_1 \wedge N_2)_{(L)}) \subset N_{2(L)}$, hence

$$\boldsymbol{i}_{\boldsymbol{z}}((N_1 \wedge N_2)_{(L)}) \subset N_{1(L)} \wedge N_{2(L)} = (N_1 \wedge N_2)_{(L)},$$

which shows $N_1 \wedge N_2$ is normal; on the other hand $N_{1(L)} = i_x(N_{1(L)}) \subset i_x((N_1 \vee N_2)_{(L)})$ and similarly $N_{2(L)} \subset i_x((N_1 \vee N_2)_{(L)})$, hence $(N_1 \vee N_2)_{(L)} \subset i_x((N_1 \vee N_2)_{(L)})$ and as these two groups have same dimension, they coincide, which proves $N_1 \vee N_2$ is normal in G.

As mentioned above, even if N_1 and N_2 are normal subgroups, $\mathfrak{N}_1 \cap \mathfrak{N}_2$ will not in general be a subhyperalgebra. But we have the surprisingly simple:

THEOREM 3. *If H is a subgroup of G, N a normal subgroup of G, then the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{H} \cup \mathfrak{N}$ is the subhyperalgebra corresponding to $H \vee N$.*

We consider, for each $r \geqq 0$, the subalgebra $\mathfrak{t}_r$ of $\mathfrak{g}_r$ generated by the union of $\mathfrak{g}_r \cap \mathfrak{H}$ and $\mathfrak{g}_r \cap \mathfrak{N}$, and the *Lie-p-algebra* $\mathfrak{m}_r$ generated by the union of $\mathfrak{t}_{r-1}$, $\mathfrak{g}_r \cap \mathfrak{H}$ and $\mathfrak{g}_r \cap \mathfrak{N}$. It is clear that $\mathfrak{t}_r$ is the subalgebra generated by $\mathfrak{m}_r$, by condition 2° of Theorem 1.

Next, consider the increasing sequence of sub-Lie-p-algebras $\mathfrak{m}_r \cap \mathfrak{g}_0$. It is a stationary sequence, in other words there is a smallest integer t such that $\mathfrak{m}_{r+1} \cap \mathfrak{g}_0 = \mathfrak{m}_r \cap \mathfrak{g}_0$ for $r \geqq t$. By a linear isomorphism, we can suppose that the basis $(X_{0i})_{1 \leqq i \leqq n}$ of $\mathfrak{g}_0$ is such that there is an increasing sequence of intervals $I = J_{-1} \subset J_0 \subset J_1 \subset \cdots \subset J_t = J_{t+1} = \cdots$ having the property that the X_{0i} with $i \in I$ form a basis for $\mathfrak{g}_0 \cap \mathfrak{N}$, the X_{0i} such that $i \in J_r$ a basis for $\mathfrak{m}_r \cap \mathfrak{g}_0$ for each value of $r \geqq 0$. As $\mathfrak{g}_0 \cap \mathfrak{N}$ is a p-ideal in the Lie algebra $\mathfrak{g}_0$, it is immediately verified that $\mathfrak{m}_0 = (\mathfrak{g}_0 \cap \mathfrak{N}) + (\mathfrak{g}_0 \cap \mathfrak{H})$.

As a basis for an inductive argument, we will now suppose that G verifies the following conditions:

(B_r) *The group law of G is pseudo-canonical and the basis (X_α) of $\mathfrak{G}$ is such that:*

a) *The X_α for which $h(\alpha) \leqq r$ and $\alpha \in \mathbf{N}^I$ form a basis for $\mathfrak{g}_r \cap \mathfrak{N}$.*

b) *The algebra $\mathfrak{t}_r$ has a basis consisting of the X_α of height $h(\alpha) \leqq r$ which satisfy the following conditions:*

($*$)$_r$ *If $\alpha = (\alpha_1, \cdots, \alpha_n)$, then $\alpha_i = 0$ for $i \notin J_r$, $\alpha_i < p^{r+1-k}$ for $i \in J_k - J_{k-1}$, $0 \leqq k \leqq r$, and $\alpha_i < p^{r+1}$ for $i \in I = J_{-1}$.*

c) *The Lie algebra $\mathfrak{m}_r$ is identical to $(\mathfrak{g}_r \cap \mathfrak{N}) + (\mathfrak{g}_r \cap \mathfrak{H}) + \mathfrak{t}_{r-1}$.*

Observe that from Lemma 1 and the fact that $\mathfrak{g}_r \cap \mathfrak{N}$ (resp. $\mathfrak{t}_r$) is a subalgebra, it follows that the Z_α such that $h(\alpha) \leqq r$ and $\alpha \in \mathbf{N}^I$ form a basis of $\mathfrak{g}_r \cap \mathfrak{N}$, and the Z_α satisfying ($*$)$_r$ and such that $h(\alpha) \leqq r$ a basis of $\mathfrak{t}_r$. We note also that the last remark before (B_r) means that G verifies conditions (B_0).

Let $\mathfrak{u}_{r+1}$ be the vector space consisting of the elements $X \in \mathfrak{g}_{r+1}$ such that $X^0 - 1 \otimes X - X \otimes 1 \in \mathfrak{t}_r \otimes \mathfrak{t}_r$. Let $\mathfrak{n}$ be the space having as a basis the $X_{r+1,i}$ of index $i \in I = J_{-1}$, $\mathfrak{q}_k$ the space having as a basis the $X_{r+1-k,i}$ of index $i \in J_k - J_{k-1}$ $(0 \leqq k \leqq r)$, $\mathfrak{r}$ the space having as a basis the X_{0i} for $i \notin J_r$. Then it is immediate to check that:

(i) *$\mathfrak{u}_{r+1}$ is the direct sum of $\mathfrak{t}_r$, $\mathfrak{n}$, the $\mathfrak{q}_k$ $(0 \leqq k \leqq r)$ and $\mathfrak{r}$.*

Next we show that:

(ii) *If $X \in \mathfrak{u}_{r+1}$ and $T \in \mathfrak{s}_r \cap \mathfrak{N}$, then $[X, T] \in \mathfrak{t}_r$.*

As $\mathfrak{u}_{r+1} \subset \mathfrak{g}_{r+1}$, $[X, T] \in \mathfrak{s}_r$. It is enough of course to take $T = Z_\alpha$ with $h(\alpha) \leqq r$ and $\alpha \in \boldsymbol{N}^I$; we use induction on $|\alpha|$. For $|\alpha| = 1$, we have $T = X_{0i}$ with $i \in I$; by definition $X^0 = 1 \otimes X + X \otimes 1 + \sum_j U_j \otimes V_j$, with U_j, V_j in $\mathfrak{t}_r$, and on the other hand $X^0{}_{0i} = 1 \otimes X_{0i} + X_{0i} \otimes 1$; hence

$$[X, X_{0i}]^0 = [X^0, X^0{}_{0i}] = 1 \otimes [X, X_{0i}] + [X, X_{0i}] \otimes 1 + \sum_j U_j \otimes [V_j, X_{0i}] + \sum_j [U_j, X_{0i}] \otimes V_j.$$

Now, as $X_{0i} \in \mathfrak{t}_r$, so are $[U_j, X_{0i}]$ and $[V_j, X_{0i}]$, hence we see that $[X, X_{0i}]$ belongs to $\mathfrak{u}_{r+1}$. But on the other hand, $[X, X_{0i}]$ belongs to the normal ideal $\mathfrak{a}$ generated by $\mathfrak{N}^+$ in $\mathfrak{G}$; but from (i) it follows at once (remembering that $\mathfrak{a}$ is also the right ideal generated by $\mathfrak{N}^+$) that $\mathfrak{s}_r \cap \mathfrak{u}_{r+1} \cap \mathfrak{a}$ is contained in $\mathfrak{t}_r$.

Next assume we have proved that $[X, Z_\lambda] \in \mathfrak{t}_r$ for $\lambda < \alpha$; then we have by definition $Z^0{}_\alpha = 1 \otimes Z_\alpha + Z_\alpha \otimes 1 + \sum_{0<\beta<\alpha} Z_\beta \otimes Z_{\alpha-\beta}$; hence

$$[X, Z_\alpha]^0 = [X^0, Z^0{}_\alpha] = 1 \otimes [X, Z_\alpha] + [X, Z_\alpha] \otimes 1 + \sum_{0<\beta<\alpha} Z_\beta \otimes [X, Z_{\alpha-\beta}] + \sum_{0<\beta<\alpha} [X, Z_\beta] \otimes Z_{\alpha-\beta} + \sum_j \sum_{0 \leqq \beta \leqq \alpha} (U_j Z_\beta \otimes V_j Z_{\alpha-\beta} - Z_\beta U_j \otimes Z_{\alpha-\beta} V_j).$$

But as $[X, Z_\beta] \in \mathfrak{t}_r$ for $\beta < \alpha$ by the induction hypothesis, we first conclude as above that $[X, Z_\alpha] \in \mathfrak{u}_{r+1}$, and then that $[X, Z_\alpha] \in \mathfrak{t}_r$. Our next step consists first in defining, by the same process as in Theorem 1, for each $i \in I$, an element $Y_{r+1,i}$ belonging to $\mathfrak{g}_{r+1} \cap \mathfrak{N}$ and such that $Y_{r+1,i} = X_{r+1,i} + \sum_{j=1}^{n} a_{ij} X_{0j}$. On the other hand, for $i \in J_k - J_{k-1}$ $(0 \leqq k \leqq r)$ we can write $X_{r-k,i} = S + T + U$, with $S \in \mathfrak{g}_r \cap \mathfrak{H}$, $T \in \mathfrak{g}_r \cap \mathfrak{N}$, $U \in \mathfrak{t}_{r-1}$. Let $S' \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$, $T' \in \mathfrak{g}_{r+1} \cap \mathfrak{N}$, $U' \in \mathfrak{t}_r$ be such that $\boldsymbol{p}'(S') = S$, $\boldsymbol{p}'(T') = T$, $\boldsymbol{p}'(U') = U$. By definition, $Y_{r-k+1,i} = S' + T' + U'$ is in $\mathfrak{u}_{r+1}$, and if we write that $\boldsymbol{p}'(Y_{r-k+1,i}) = X_{r-k,i}$, we see at once that we can write $Y_{r-k+1,i} = X_{r-k+1,i} + \sum_{j=1}^{n} a_{ij} X_{0j} + V$

with $V \in \mathfrak{t}_r$; we thus see that, by substracting V from U', we can find an element $Y_{r-k+1,i}$ in $(\mathfrak{g}_{r+1} \cap \mathfrak{H}) + (\mathfrak{g}_{r+1} \cap \mathfrak{N}) + \mathfrak{t}_r$, of the form $X_{r-k+1,i} + \sum_{j=1}^{n} a_{ij} X_{0j}$. It is clear that all the Y_{hi} thus defined are linearly independent mod $\mathfrak{t}_r$; if $\mathfrak{v}$ is the space having these Y_{hi} as a basis, $\mathfrak{v}$ is contained in $(\mathfrak{g}_{r+1} \cap \mathfrak{H}) + (\mathfrak{g}_{r+1} \cap \mathfrak{N}) + \mathfrak{t}_r$, and $\mathfrak{u}_{r+1}$ is the direct sum $\mathfrak{t}_r + \mathfrak{v} + \mathfrak{r}$; therefore we can find a certain number of elements $Y_{0i} \in \mathfrak{r}$ $(1 \leqq i \leqq q)$ which form a basis of a supplementary subspace of $\mathfrak{v} + \mathfrak{t}_r$ in $(\mathfrak{g}_{r+1} \cap \mathfrak{N}) + (\mathfrak{g}_{r+1} \cap \mathfrak{H}) + \mathfrak{t}_r$.

(iii) *If* $T \in \mathfrak{g}_{r+1} \cap \mathfrak{N}$ *and* $U \in \mathfrak{t}_r$, *then* $[T, U] \in \mathfrak{t}_r$.

This is trivial if $T \in \mathfrak{s}_r \cap \mathfrak{N} \subset \mathfrak{t}_r$, and we can thus only consider the case $T = Y_{r+1,i}$, $i \in I$. As $Y_{r+1,i} \in \mathfrak{u}_{r+1}$, $Y^0_{r+1,i} - 1 \otimes Y_{r+1,i} - Y_{r+1,i} \otimes 1$ belongs to $\mathfrak{t}_r \otimes \mathfrak{t}_r$. We can on the other hand take $U = Z_\alpha$ with α satisfying $(*)_r$, and then prove (iii) by induction on $|\alpha|$; the proof follows exactly the same lines as that of (ii), and we omit it.

(iv) *If* $T \in \mathfrak{g}_{r+1} \cap \mathfrak{N}$, $U \in \mathfrak{u}_{r+1}$, *then* $[T, U] \in (\mathfrak{g}_{r+1} \cap \mathfrak{N}) + \mathfrak{t}_r$.

Due to (ii), we can suppose $T = Y_{r+1,i}$, $i \in I$; then the proof again follows the same lines as (ii), using the fact that $Y^0_{r+1,i} - 1 \otimes Y_{r+1,i} - Y_{r+1,i} \otimes 1$ is in $(\mathfrak{s}_r \cap \mathfrak{N}) \otimes (\mathfrak{s}_r \cap \mathfrak{N})$, the results (ii) and (iii), and noting that $\mathfrak{u}_{r+1} \cap \mathfrak{a} \subset (\mathfrak{g}_{r+1} \cap \mathfrak{N}) + \mathfrak{t}_r$.

(v) *The Lie-p-algebra* $\mathfrak{m}_{r+1}$ *is identical to* $(\mathfrak{g}_{r+1} \cap \mathfrak{N}) + (\mathfrak{g}_{r+1} \cap \mathfrak{H}) + \mathfrak{t}_r$.

By definition of $\mathfrak{m}_r$, it is enough to prove that $\mathfrak{m}'_{r+1} = (\mathfrak{g}_{r+1} \cap \mathfrak{N}) + (\mathfrak{g}_{r+1} \cap \mathfrak{H}) + \mathfrak{t}_r$ is a Lie-p-algebra. For any two elements X, X' of $\mathfrak{m}'_{r+1}$, we have $X + T + U \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$, $X' + T' + U' \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$ for suitable elements T, T' in $\mathfrak{g}_{r+1} \cap \mathfrak{N}$, U, U' in $\mathfrak{t}_r$. As $[X + T + U, X' + T' + U']$ is in $\mathfrak{g}_{r+1} \cap \mathfrak{H}$, X, X' in $\mathfrak{u}_{r+1}$, it follows from (ii) and (iv) that $[X, X']$ belongs to $\mathfrak{m}'_{r+1}$. Similarly $(X + T + U)^p \in \mathfrak{g}_{r+1} \cap \mathfrak{H}$; but by Jacobson's formula, $(X + T + U)^p$ is the sum of $X^p + T^p + U^p$ and of alternants formed with X, T and U; we have just seen that the latter are in $\mathfrak{m}'_{r+1}$, and as $U^p \in \mathfrak{t}_r$ and $T^p \in \mathfrak{g}_{r+1} \cap \mathfrak{N}$, we have again $X^p \in \mathfrak{m}'_{r+1}$, which concludes the proof of (v).

This shows that q is the number of elements of $J_{r+1} - J_r$, and that the Y_{0i} are linear combinations of the X_{0j} of index $j \in J_{r+1} - J_r$, let $Y_{0i} = \sum_{j \in J_{r+1} - J_r} b_{ij} X_{0j}$; let (c_{ij}) be the inverse matrix to (b_{ij}). Consider the isomorphism $\boldsymbol{u}$ of G onto a group G' defined by

$$u_i(\boldsymbol{x}) = h_i(\boldsymbol{x}) - \sum_{k=-1}^{r} \Big(\sum_{j \in J_k - J_{k-1}} a_{ji} x^{p^{r-k+1}}_j \Big) \qquad \text{(taking } J_{-2} = \varnothing\text{)}$$

where $h_i(\boldsymbol{x}) = x_i$ if $i \notin J_{r+1} - J_r$, and $h_i(\boldsymbol{x}) = \sum_{j \in J_{r+1}-J_r} c_{ij}x_j$ if $i \in J_{r+1} - J_r$.

Then it follows readily from the definitions [9, p. 221] that

$$\begin{aligned} &\boldsymbol{u}'(Z_\alpha) = Z'_\alpha \text{ for indices } \alpha \text{ verifying } (*)_r \\ &\boldsymbol{u}'(Y_{r-k+1,i}) = X'_{r-k+1,i} \text{ for } i \in J_k - J_{k-1}, \qquad -1 \leqq k \leqq r+1 \\ &\boldsymbol{u}'(X_{0i}) = X'_{0i} \text{ for } i \notin J_{r+1} - J_r. \end{aligned} \tag{12}$$

The group law of G' is now not pseudo-canonical any more, and we have to apply to G' the isomorphism $\boldsymbol{v}$ onto a group G'' obtained by the "standard process" of no. 3. Now, if $\boldsymbol{f}$ is the homomorphism of $\mathfrak{H}_n(K)$ onto $\mathfrak{G}$ such that $\boldsymbol{f}(V_\alpha) = Z_\alpha$ for every index α, it follows from (12) that $\boldsymbol{u}'(\boldsymbol{f}(V_\alpha)) = Z'_\alpha$ for all α verifying $(*)_r$. As explained in no. 3, $\boldsymbol{v}$ is obtained as an "infinite product" $\cdots \boldsymbol{v}_h \boldsymbol{v}_{h-1} \cdots \boldsymbol{v}_0$; using the preceding remark and the recursive definition of the $\boldsymbol{v}_h$, it is easily verified by induction that for any h, $\boldsymbol{v}'_h(Z^{(h)}_\alpha) = Z^{(h+1)}_\alpha$ for all α satisfying $(*)_r$ and in addition

$$\boldsymbol{v}'_h(X^{(h)}_{r-k+1,i}) = X^{(h+1)}_{r-k+1,i} \text{ for } i \in J_k - J_{k-1}, \qquad -1 \leqq k \leqq r+1,$$

as well as $\boldsymbol{v}'_h(X^{(h)}_{0i}) = X^{(h+1)}_{0i}$ for $1 \leqq i \leqq n$ (with self-explanatory notations). We suppress the somewhat tedious detail, which involves not the slightest difficulty, the main point being the following one: if, in an arbitrary power series without constant term, we substitute for the indeterminates power series in the x_i $(1 \leqq i \leqq n)$, in which there is no monomial in $\boldsymbol{x}^\alpha$ with α satisfying $(*)_r$, or $\alpha = p^{r-k+1}\epsilon_i$, $i \in J_k - J_{k-1}$ $(-1 \leqq k \leqq r+1)$, or $\alpha = \epsilon_i$ $(1 \leqq i \leqq n)$ then there is also no monomial of any of these types in the power series which results from the substitution. The same remark shows that if we consider the isomorphism $\boldsymbol{w} = \boldsymbol{v} \circ \boldsymbol{u}$ of G onto G'', there are no monomials in $\boldsymbol{x}^\alpha$ with α satisfying $(*)_r$ in any of the series $w_i(\boldsymbol{x}) - x_i$.

Replacing for a moment G by G'', we have obtained as a basis for $\mathfrak{m}_{r+1}$ the X_α which satisfy $(*)_r$ and the $X_{r-k+1,i}$ such that $i \in J_k - J_{k-1}$ $(-1 \leqq k \leqq r+1)$. As $\mathfrak{m}_{r+1}$ is a Lie-p-algebra, it is immediate to see that the associative algebra $\mathfrak{t}_{r+1}$ it generates has as a basis the X_α which satisfy $(*)_{r+1}$. In other words, we have obtained a group for which (B_{r+1}) holds.

Summing up what we have done, and changing the notations, we have proved the following result: if G_r is a group, H_r a subgroup and N_r a normal subgroup of G_r for which assumptions (B_r) are satisfied, then there is an isomorphism $\boldsymbol{g}_r$ of G_r onto a group G_{r+1} such that $H_{r+1} = \boldsymbol{g}_r(H_r)$ and $N_{r+1} = \boldsymbol{g}_r(N_r)$ now satisfy assumptions (B_{r+1}). Moreover, as soon as $r > t$, in the power series $g_{ri}(\boldsymbol{x}) - x_i$ $(1 \leqq i \leqq n)$ there are no monomials of height $< r - t$. We conclude as usual (see end of proof of Theorem 1), obtaining

an isomorphism $\boldsymbol{g}$ of G onto a group $\bar{G}$ such that if $\bar{H} = \boldsymbol{g}(H)$ and $\bar{N} = \boldsymbol{g}(N)$ $\bar{H}$ and $\bar{N}$ now satisfy assumptions (B_r) for *every* r. The subalgebra of $\bar{\mathfrak{G}}$ generated by $\bar{\mathfrak{H}} \cup \bar{\mathfrak{N}}$ is then the algebra having as a basis the $\bar{Z}_\alpha$ such that $\alpha \in N^{\prime\prime}$, and it is clear that it is a typical subhyperalgebra; the proof of Theorem 3 is thus complete.

From Corollary 2 to Proposition 9 and the chain conditions, it follows that the normal subgroups of G form a *complete lattice*. There is a corresponding result for the normal ideals of $\mathfrak{G}$:

PROPOSITION 10. *The normal ideals of $\mathfrak{G}$ form a complete lattice; more precisely, the mapping $\mathfrak{N} \to \mathfrak{a}(\mathfrak{N})$ which, to every subhyperalgebra $\mathfrak{N}$ corresponding to a normal subgroup of G, associates the normal ideal it generates, is a lattice isomorphism, such that* $\mathfrak{a}(\mathfrak{N}_1 \vee \mathfrak{N}_2) = \mathfrak{a}(\mathfrak{N}_1) + \mathfrak{a}(\mathfrak{N}_2)$.

Indeed, by Theorem 3, $\mathfrak{a}(\mathfrak{N}_1) + \mathfrak{a}(\mathfrak{N}_2)$ is the ideal generated by the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{N}_1 \cup \mathfrak{N}_2$, hence by $\mathfrak{N}_1 \vee \mathfrak{N}_2$. It is clear that the condition $\mathfrak{N}_1 \subset \mathfrak{N}_2$ implies $\mathfrak{a}(\mathfrak{N}_1) \subset \mathfrak{a}(\mathfrak{N}_2)$; conversely, if that last inclusion holds, then $\mathfrak{a}(\mathfrak{N}_1 \vee \mathfrak{N}_2) = \mathfrak{a}(\mathfrak{N}_2)$. If we had $\mathfrak{N}_1 \vee \mathfrak{N}_2 \neq \mathfrak{N}_2$, the dimension of $N_1 \vee N_2$ would be strictly greater than that of N_2; but the dimension of a normal subgroup N is equal to $\dim(\mathfrak{g}_0 \cap \mathfrak{a}(\mathfrak{N}))$, hence our conclusion.

10. The isomorphism theorems and the Jordan-Hölder theorem. For convenience, we collect the properties corresponding to the classical "isomorphism theorems" of ordinary group theory:

PROPOSITION 11. *Let $\boldsymbol{u}$ be a homomorphism of G onto a group G', having kernel $N = \boldsymbol{u}^{-1}(e)$. Then:*

a) *The mapping $H \to \boldsymbol{u}(H)$ is a bijection of the set of subgroups of G containing N, onto the set of subgroups of G', the inverse bijection being $H' \to \boldsymbol{u}^{-1}(H')$.*

b) *In the preceding one-to-one correspondence, to normal subgroups correspond normal subgroups, and the correspondence is a lattice isomorphism. Moreover, if $H \supset N$ is normal in G, and $H' = \boldsymbol{u}(H)$, there is an isogeny of G/H onto G'/H', and both groups are isomorphic when $\boldsymbol{u}$ is an epimorphism.*

c) *If F is an arbitrary subgroup of G, $F \vee N = \boldsymbol{u}^{-1}(\boldsymbol{u}(F))$, there exist isogenies of $F/(F \wedge N)$ onto $(F \vee N)/N$ and of $(F \vee N)/N$ onto $\boldsymbol{u}(F)$. Moreover,* $\dim F + \dim N = \dim(F \vee N) + \dim(F \wedge N)$.

Most of this has already been proved (Propositions 8 and 9). The fact

that if H is normal in G, $\boldsymbol{u}(H)$ is normal in G' follows from the characterization of the subhyperalgebras of normal subgroups given in Theorem 2, remembering that $\boldsymbol{u}'(\mathfrak{G}) = \mathfrak{G}'$. Conversely, if H' is normal in G', and $\boldsymbol{v}$ is the natural epimorphism of G' onto G'/H', $\boldsymbol{v} \circ \boldsymbol{u}$ is a homomorphism of G onto G'/H', of kernel $H = \boldsymbol{u}^{-1}(H')$, hence H is normal in G and there is an isogeny of G/H onto G'/H' (Proposition 6); this isogeny is an isomorphism when $\boldsymbol{u}$ is an epimorphism, since then $\boldsymbol{v} \circ \boldsymbol{u}$ is an epimorphism (Corollary to Proposition 1). The isogenies of $(F \vee N)/N$ and $F/(F \wedge N)$ onto $\boldsymbol{u}(F)$ stem from application of Proposition 6 to the restrictions of $\boldsymbol{u}$ to $F \vee N$ and F respectively. If we replace $\boldsymbol{u}$ by the natural epimorphism of G onto G/N, it follows from Proposition 1 that the restriction of $\boldsymbol{u}$ to $F \vee N$ is an epimorphism onto $\boldsymbol{u}(F)$; in that case $\boldsymbol{u}(F)$ and $(F \vee N)/N$ are isomorphic, hence the existence of an isogeny of $F/(F \wedge N)$ onto $(F \vee N)/N$ (cf. no. 21).

COROLLARY. *Under the same assumptions as in Proposition* 11, *the mapping* $\mathfrak{a}(\mathfrak{H}) \to \boldsymbol{u}'(\mathfrak{a}(\mathfrak{H}))$ *is a lattice isomorphism of the set of normal ideals* $\mathfrak{a}(\mathfrak{H})$ *corresponding to normal subgroups* H *of* G *containing* N, *onto the set of all normal ideals of* $\mathfrak{G}'$, *the inverse mapping being* $\mathfrak{a}(\mathfrak{H}') \to \boldsymbol{u}'^{-1}(\mathfrak{a}(\mathfrak{H}'))$.

It is now possible to develop the sequence of propositions and definitions corresponding to the one which, for abstract groups (or Lie groups over fields of characteristic 0), leads to the *Jordan-Hölder theorem*: namely the Zassenhaus lemma, the definition of simple groups[7a] and the Schreier refinement theorem. The only difference is that throughout "*isomorphic groups*" are to be replaced by "*isogenous groups*"; "equivalent" composition series are those in which the quotient groups can be paired so as to be *isogenous*; and the Jordan-Hölder theorem states then that two Jordan-Hölder series are equivalent. We suppress the details, which are straightforward adaptations of the classical proofs (see e.g. [3, pp. 84-87], and [16, pp. 422-423], where the same thing is done for algebraic groups); as we have not yet established the lattice property of subgroups, the only point where a little care is needed is in proving the existence of the g.l.b. $G_i \wedge H_j$ when $(G_i)_{0 \leq i \leq n}$ and $(H_j)_{0 \leq j \leq m}$ are two composition series of G. This is done for instance by induction on j, remarking that H_{j+1} is normal in H_j, hence that $(G_i \wedge H_j) \wedge H_{j+1}$ is defined by Proposition 9. It should be observed that this is the best possible statement, in other words that the Jordan-Hölder theorem in its usual form *does not hold* for formal Lie groups over a field of characteristic $p>0$ (see no. 21).

[7a] One must beware of the fact that if G is a simple Lie group over a field K, and L a perfect extension of K, $G_{(L)}$ is not necessarily simple.

11. Commuting subgroups and center. Let (H, f) and (H', g) be two subgroups of G; we say they *commute* ("elementwise"[8]) or that they *centralize* each other if $f(y)g(y') = g(y')f(y)$ for any independent generic points y, y' of H, H' respectively.

PROPOSITION 12. *In order that two subgroups of G commute, it is necessary and sufficient that their subhyperalgebras commute elementwise in $\mathfrak{G}$.*

Let (Y_λ), (Y'_μ) and (Z_α) be the structural bases of $\mathfrak{H}$, $\mathfrak{H}'$, $\mathfrak{G}$ respectively. If we put $(f(y))^\alpha = \sum_\lambda a_{\lambda\alpha} y^\lambda$, $(g(y'))^\beta = \sum_\mu b_{\mu\beta} y'^\mu$, the relation $f(y)g(y') = g(y')f(y)$ is by definition equivalent to the relations

$$\sum_{\alpha,\beta,\lambda,\mu} d_{\alpha\beta\gamma} a_{\lambda\alpha} b_{\mu\beta} y^\lambda y'^\mu = \sum_{\alpha,\beta,\lambda,\mu} d_{\alpha\beta\gamma} b_{\mu\alpha} a_{\lambda\beta} y^\lambda y'^\mu$$

for any index γ. This yields $\sum_{\alpha,\beta} d_{\alpha\beta\gamma} a_{\lambda\alpha} b_{\mu\beta} = \sum_{\alpha,\beta} d_{\alpha\beta\gamma} b_{\mu\alpha} a_{\lambda\beta}$ for all λ, μ. But we have $f'(Y_\lambda) = \sum_\alpha a_{\lambda\alpha} Z_\alpha$ and $g'(Y'_\mu) = \sum_\alpha b_{\mu\alpha} Z_\alpha$, hence the preceding relations are equivalent to $f'(Y_\lambda)g'(Y'_\mu) = g'(Y'_\mu)f'(Y_\lambda)$, q. e. d.

Proposition 12 and Theorem 2 show immediately that any subgroup of G which centralizes G is a *normal* subgroup of G; if Z_1, Z_2 are any such subgroups, $Z_1 \vee Z_2$ is therefore defined and a normal subgroup of G; and it follows from Proposition 12 and Theorem 3 that $Z_1 \vee Z_2$ again centralizes G. There is therefore a *largest* subgroup centralizing G, the *center* Z (or $Z(G)$) of G, which is an abelian normal subgroup. It follows from Proposition 12 that the subhyperalgebra $\mathfrak{Z}$ of the center Z is *the largest subhyperalgebra contained in the center* of the algebra $\mathfrak{G}$.

12. Quasi-direct products. We say that a group G is the *quasi-direct product* of two of its subgroups H_1, H_2, if there exists an *isogeny* u of a direct product $G_1 \times G_2$ onto G such that $u(G_1) = H_1$, $u(G_2) = H_2$.

PROPOSITION 13. *In order that a group G be quasi-direct product of two subgroups H_1, H_2, necessary and sufficient conditions are that H_1 and H_2 be normal subgroups such that $H_1 \wedge H_2 = e$ and $H_1 \vee H_2 = G$; when these conditions are satisfied, H_1 and H_2 commute.*

The necessity of the conditions follows at once from Proposition 7. Conversely, suppose these conditions are satified; it follows from Propositions

[8] I have not been able to find, for formal Lie groups, the notion corresponding to "set-theoretic" commutation of two subgroups in the classical theory (i. e. the relation $HH' = H'H$).

11 and 2 that there is a homomorphism $\boldsymbol{v}_1$ (resp. $\boldsymbol{v}_2$) of G onto a group G_1 (resp. G_2) isogenous to H_1 (resp. H_2), with kernel H_2 (resp. H_1). Consider the homomorphism $\boldsymbol{v}=(\boldsymbol{v}_1,\boldsymbol{v}_2)$ of G into $G_1\times G_2$; we have $\boldsymbol{v}(H_1)=G_1$ and $\boldsymbol{v}(H_2)=G_2$, hence $\boldsymbol{v}(G)$ contains $G_1\vee G_2=G_1\times G_2$; $\boldsymbol{v}$ is thus surjective, and as G and $G_1\times G_2$ have same dimension by Proposition 11, $\boldsymbol{v}$ is an *isogeny*. Let $\boldsymbol{w}$ be an isogeny of $G_1\times G_2$ onto some group $G^{(t)}$ (prop. 2), such that $\boldsymbol{w}\circ\boldsymbol{v}=\boldsymbol{p}^t$; $\boldsymbol{w}^{(-t)}$ is then an isogeny of $G_1^{(-t)}\times G_2^{(-t)}$ onto G, mapping $G_1^{(-t)}$ onto H_1 and $G_2^{(-t)}$ onto H_2. The last statement of the proposition is obvious from the definition of a quasi-direct product.

The preceding definition and proof can of course be extended in an obvious way to any finite number of factors. For an example of a quasi-direct product which is not a direct product, see no. 21.

13. Subgroup of commutators; ascending and descending central series. Let N_1,N_2 be two normal subgroups of G, $\boldsymbol{u}$ a homomorphism of G into a group H, P the kernel of $\boldsymbol{u}$. In order that the subgroups $\boldsymbol{u}(N_1)$ and $\boldsymbol{u}(N_2)$ commute in H, the corresponding subhyperalgebras of $\mathfrak{H}$ must commute elementwise (prop. 12). Using Propositions 6 and 7, we have an equivalent condition by restricting ourselves to the case in which $\boldsymbol{u}$ is the natural epimorphism $\boldsymbol{g}$ of G onto G/P. But then the subhyperalgebra corresponding to $\boldsymbol{g}(N_1)$ (resp. $\boldsymbol{g}(N_2)$) consists of the classes of the elements of $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$) modulo the normal ideal $\mathfrak{a}(\mathfrak{P})$ (th. 2); hence, for $T_1\in\mathfrak{N}_1$, $T_2\in\mathfrak{N}_2$, $[T_1,T_2]$ must belong to $\mathfrak{a}(\mathfrak{P})$, and conversely.

PROPOSITION 14. *If N_1,N_2 are two normal subgroups*[9] *of G, there exists a smallest normal subgroup $P=[N_1,N_2]$ of G such that the natural images of N_1 and N_2 in G/P commute; $[N_1,N_2]$ is contained in $N_1\wedge N_2$, and $\mathfrak{a}(\mathfrak{P})$ is the smallest normal ideal containing the alternants $[T_1,T_2]$ where $T_1\in\mathfrak{N}_1$ and $T_2\in\mathfrak{N}_2$.*

Once the existence of a smallest P is proved, the characterization of $\mathfrak{a}(\mathfrak{P})$ becomes obvious. As N_1 and N_2 are normal subgroups, $[T_1,T_2]$ belongs both to $\mathfrak{a}(\mathfrak{N}_1)$ and $\mathfrak{a}(\mathfrak{N}_2)$, hence $P\subset N_1\wedge N_2$ by Proposition 10. Finally, owing to the chain conditions, all we have to prove is that if P_1,P_2 are kernels of homomorphisms $\boldsymbol{g}_1,\boldsymbol{g}_2$ of G into groups H_1,H_2, such that $\boldsymbol{g}_i(N_1)$ and $\boldsymbol{g}_i(N_2)$ commute ($i=1,2$), then $P_1\wedge P_2$ has the same property. But $P_1\wedge P_2$ is the kernel of the homomorphism $\boldsymbol{g}=(\boldsymbol{g}_1,\boldsymbol{g}_2)$ of G into $H_1\times H_2$, and it is clear that $\boldsymbol{g}(N_1)$ and $\boldsymbol{g}(N_2)$ commute in $H_1\times H_2$.

[9] There does not seem to be any obvious definition of $[H_1,H_2]$ when H_1,H_2 are arbitrary subgroups of a formal Lie group G.

We will denote by $[\mathfrak{N}_1, \mathfrak{N}_2]$ the hyperalgebra of $[N_1, N_2]$. If $\boldsymbol{g}$ is the natural epimorphism of G onto $G/[N_1, N_2]$, then $\boldsymbol{g}(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}^{-1}{}_1\boldsymbol{x}^{-1}{}_2) = \boldsymbol{e}$ for any two (independent) generic points $\boldsymbol{x}_1$ of N_1, $\boldsymbol{x}_2$ of N_2; if $[N_1, N_2]$ is brought into typical form, being defined by the annulation of all coordinates of index $i > m$, then the coordinates of such indices of $\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}^{-1}{}_1\boldsymbol{x}^{-1}{}_2$ are 0.

In particular, for any normal subgroup N of G, $[N, N]$ is the smallest normal subgroup *in* G such that $N/[N, N]$ be *abelian*. We call $[G, G]$ *the commutator subgroup* or *derived subgroup* of G, and write it $D(G)$ (and its subhyperalgebra $D(\mathfrak{G})$).

PROPOSITION 15. *For any normal subgroup N of G, $[N, N] = D(N)$.*

All we have to show is that $D(N)$ is a normal subgroup *in* G. Now, if $\boldsymbol{z}$ is a generic point of G, and L a perfect extension of $K[[\boldsymbol{z}]]$, $\boldsymbol{i}_{\boldsymbol{z}}$ is an automorphism of $N_{(L)}$, hence $H = \boldsymbol{i}_{\boldsymbol{z}}(D(N_{(L)}))$ is a normal subgroup of $N_{(L)}$ such that $N_{(L)}/H$ is abelian. By definition, this implies $H \supset D(N_{(L)}))$, and as the two groups have same dimension, they are equal.

We can define as usual the *derived series* of successive commutator subgroups $D^n(G) = D(D^{n-1}(G))$ for $n > 1$, and the *descending central series* $C^1(G) = D(G)$, $C^n(G) = [G, C^{n-1}(G)]$ for $n > 1$.

PROPOSITION 16. *If $\boldsymbol{u}$ is a surjective homomorphism of G onto G', and N_1, N_2 two normal subgroups of G, then $[\boldsymbol{u}(N_1), \boldsymbol{u}(N_2)] = \boldsymbol{u}([N_1, N_2])$.*

Indeed, if $\mathfrak{a} = \mathfrak{a}([\mathfrak{N}_1, \mathfrak{N}_2])$ the image $\boldsymbol{u}'(\mathfrak{a})$ is a normal ideal in $\mathfrak{G}'$ containing all the $[\boldsymbol{u}'(T_1), \boldsymbol{u}'(T_2)] = \boldsymbol{u}'([T_1, T_2])$ for $T_1 \in \mathfrak{N}_1$, $T_2 \in \mathfrak{N}_2$. Conversely, if $\mathfrak{a}' \subset \mathfrak{G}'$ is a normal ideal containing these elements, $\boldsymbol{u}'^{-1}(\mathfrak{a}')$ is a normal ideal in $\mathfrak{G}$ containing the $[T_1, T_2]$, hence contains $\mathfrak{a}$ by definition, and therefore $\boldsymbol{u}'(\mathfrak{a})$ is the smallest normal ideal in $\mathfrak{G}'$ containing the $[\boldsymbol{u}'(T_1), \boldsymbol{u}'(T_2)]$.

COROLLARY. *If $\boldsymbol{u}$ is a surjective homomorphism of G onto G', then $\boldsymbol{u}(D^n(G)) = D^n(G')$, $\boldsymbol{u}(C^n(G)) = C^n(G')$.*

PROPOSITION 17. *Let M_1, M_2 be two normal subgroups of G, H a subgroup of G, N_1, N_2 two normal subgroups of H such that $N_1 \subset M_1$, $N_2 \subset M_2$. Then $[N_1, N_2] \subset [M_1, M_2]$.*

Indeed, if $\boldsymbol{g}$ is the natural mapping of G onto $G/[M_1, M_2]$, $\boldsymbol{g}(M_1)$ and $\boldsymbol{g}(M_2)$ commute, hence also $\boldsymbol{g}(N_1) \subset \boldsymbol{g}(M_1)$ and $\boldsymbol{g}(N_2) \subset \boldsymbol{g}(M_2)$; the kernel $H \wedge [M_1, M_2]$ of the restriction of $\boldsymbol{g}$ to H thus contains $[N_1, N_2]$.

COROLLARY. *For any subgroup H of G,*

$$D^n(H) \subset D^n(G) \text{ and } C^n(H) \subset C^n(G).$$

A group G will be called *solvable* (resp. *nilpotent*) if $D^n(G)=e$ (resp. $C^n(G)=e$) for sufficiently large n. From Propositions 16 and 17 it follows that subgroups and quotient groups of solvable (resp. nilpotent) groups are solvable (resp. nilpotent). The usual arguments show that solvable groups are characterized as those having a composition series with abelian quotient groups, and that if H is solvable and normal in G, and G/H solvable, then G is solvable.

The *ascending central series* is also defined as usual: $Z^1(G)$ is the center of G and $Z^n(G)$ the inverse image of the center of $G/Z^{n-1}(G)$. As, by definition, $C^{n-1}(G)/C^n(G)$ is contained in the center of $G/C^n(G)$, the classical argument shows that nilpotent groups are also those for which the ascending central series terminates at G.

III. Representable Groups.

14. Formal Lie groups associated to algebraic groups.[10] The words "algebraic group" will be taken with the meaning given to them in [16]. The process outlined in [8, p. 116], associating a formal Lie group to an algebraic group which is an affine variety, can be generalized in an obvious way to arbitrary algebraic groups. We rapidly sketch it in this section.

We first recall known properties of *local rings* at simple points of algebraic varieties. Let U, V be two algebraic varieties, F an everywhere defined rational map of U into V, K a common field of definition of U, V, F, a a simple point of U rational over K, b a simple point of V rational over K, and suppose $b=F(a)$. Let x, y be generic points of U, V over K, and consider an element $t(y)\in K(y)$ of the local ring $\mathfrak{o}_K(b, V)$ of V at the point b. As the specialization $y\to b$ transforms $t(y)$ into a finite element $t(b)$ and the specialization $x\to a$ transforms $F(x)$ into b, the specialization $y\to F(x)$ transforms $t(y)$ into a finite $t(F(x))$, which is an element of the local ring $\mathfrak{o}_K(a, U)$, and the mapping $t(y)\to t(F(x))$ is a homomorphism of $\mathfrak{o}_K(b, V)$ into $\mathfrak{o}_K(a, U)$, which can be extended by continuity to a homomorphism $\bar{F}$ of the completed local ring $\mathfrak{O}_K(b, V)$ into the completed local ring $\mathfrak{O}_K(a, U)$. If $\dim U=m$, $\dim V=n$, then $\mathfrak{O}_K(a, U)$ (resp. $\mathfrak{O}_K(b, V)$) can be identified to the ring of power series $K[[X_1, \cdots, X_m]]$ (resp.

[10] The assumption that the characteristic is $\neq 0$ is not used in this part until no. 19 (Lemma 3 and its consequences). Taking into account the one-to-one correspondence between formal Lie groups and Lie algebras in characteristic 0, the theorems proved in nos. 14-18 yield new proofs of the corresponding theorems on Lie algebras proved by Chevalley in [6].

$K[[Y_1, \cdots, Y_n]]$) in m (resp. n) indeterminates, by an isomorphism which sends a system $(s_i)_{1 \leq i \leq m}$ (resp. $(t_j)_{1 \leq j \leq n}$) of uniformizing parameters in $\mathfrak{o}_K(a, U)$ (resp. $\mathfrak{o}_K(b, V)$) onto the system (X_i) (resp. (Y_j)); the elements $\bar{F}(t_j)$ of $\mathfrak{D}_K(a, U)$ are then identified to n power series $\phi_j(X_1, \cdots, X_m)$; if t is an element of $\mathfrak{D}_K(b, V)$, identified to a power series $f(Y_1, \cdots, Y_n)$, $\bar{F}(t)$ is identified to the power series obtained by replacing each Y_j by $\phi_j(X_1, \cdots, X_m)$ in the power series $f(Y_1, \cdots, Y_n)$.

If W is a third algebraic variety, G an everywhere defined rational map of V into W, W and G being defined over K, c a simple point of W, rational over K, such that $G(b) = c$, then it is readily verified that $\overline{G \circ F} = \bar{F} \circ \bar{G}$.

In particular, suppose F is *generically surjective*, i.e. that $F(x)$ is a generic point of V over K [16]; then it is obvious that the restriction of $\bar{F}$ to $\mathfrak{o}_K(b, V)$ is injective, and it can be proved [17, p. 64, th. 3] that $\bar{F}$ itself is *injective*.

Consider now the product $U \times V$, in which (a, b) is a simple point; to the projections pr_U and pr_V correspond natural injections of $\mathfrak{o}_K(a, U)$ and $\mathfrak{o}_K(b, V)$ into $\mathfrak{o}_K((a, b), U \times V)$; the images $s'_1, \cdots, s'_m, t'_1, \cdots, t'_n$ of the s_i and t_j by these injections form a system of $m + n$ uniformizing parameters of $U \times V$ at the point (a, b), which we will say *corresponds naturally* to the systems (s_i) and (t_j).

Finally, suppose U is a subvariety of V, $b = a$, F being the natural injection of U into V. Then $\bar{F}$ is a homomorphism of $\mathfrak{D}_K(a, V)$ *onto* $\mathfrak{D}_K(a, U)$; and it can be shown that uniformizing parameters t_j in $\mathfrak{o}_K(a, V)$ may be chosen so that the kernel of $\bar{F}$ is the ideal generated by $t_{m+1}, \cdots, t_n$, and the $\bar{F}(t_i)$ for $1 \leqq i \leqq m$ form a system of uniformizing parameters of $\mathfrak{o}_K(a, U)$ [4, p. 705, prop. 9].

Let now G be an algebraic group of dimension n defined over K, G_0 the connected component of the identity e in G. Applying the process described above to the rational mapping $f: (x, y) \to xy$ of $G_0 \times G_0$ into G_0, we obtain a homomorphism $\bar{f}$ of $\mathfrak{D}_K(e, G_0)$ into $\mathfrak{D}_K((e, e), G_0 \times G_0)$; let $(t_i)_{1 \leq i \leq n}$ be a system of uniformizing parameters in $\mathfrak{o}_K(e, G_0)$, and let $(t'_1, \cdots, t'_n, t''_1, \cdots, t''_n)$ be the system of uniformizing parameters corresponding naturally to (t_i) in $\mathfrak{o}_K((e, e), G_0 \times G_0)$; then, if (t'_i) and (t''_i) are identified to two systems $\boldsymbol{x} = (x_i)$, $\boldsymbol{y} = (y_i)$ of n indeterminates, each element $\bar{f}(t_i)$ is identified to a power series $\phi_i(\boldsymbol{x}, \boldsymbol{y})$ in these $2n$ indeterminates. Consideration of the composed mappings $(x, y, z) \to (xy, z) \to xyz$, $(x, y, z) \to (x, yz) \to xyz$ of $G_0 \times G_0 \times G_0$ into G_0, and of the composed mappings $x \to (x, e) \to xe$, $x \to (e, x) \to ex$ of G_0 into itself proves at once that the power series ϕ_i define a *group law* over K. Let us for the moment denote by $G^*((t_i))$ the

formal Lie group thus associated to G, K and (t_i); it is immediate that when K is replaced by a larger field of definition L of G, the power series ϕ_i are not changed, in other words $G^*((t_i))$ is replaced by $(G^*((t_i)))_{(L)}$.

Let now H be a second algebraic group of dimension m, e' the identity in H, H_0 its connected component, u a rational homomorphism of G into H, K a field of definition for G, H and u[11]; it is known that u, restricted to G_0, is a rational homomorphism of G_0 into the connected component H_0 of H. It gives rise therefore to a homomorphism $\bar{u}$ of $\mathfrak{O}_K(e', H_0)$ into $\mathfrak{O}_K(e, G_0)$. Let $(s_j)_{1\leq j\leq m}$ be a system of uniformizing parameters in $\mathfrak{o}_K(e', H_0)$; then, if (t_i) is identified to a system $\boldsymbol{x} = (x_i)$ of n indeterminates, each element $\bar{u}(s_j)$ is identified to a power series $u_j(\boldsymbol{x})$ in these n indeterminates; and consideration of the composed mappings $(x, y) \to (u(x), u(y)) \to u(x)u(y)$ and $(x, y) \to xy \to u(xy)$ of $G_0 \times G_0$ into H_0 shows that the system $\boldsymbol{u} = (u_j)$ of m power series is a *homomorphism* of $G^*((t_i))$ into $H^*((s_j))$. If u is a rational homomorphism of G into G', v a rational homomorphism of G' into G'', then (with self-explanatory notations) if we consider the rational homomorphism $w = v \circ u$ of G into G'', the corresponding homomorphisms $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ of $G^*((t_i))$ into $G'^*((t'_j))$, of $G'^*((t'_j))$ into $G''^*((t''_k))$ and of $G^*((t_i))$ into $G''^*((t''_k))$ are such that $\boldsymbol{w} = \boldsymbol{v} \circ \boldsymbol{u}$. If we take in particular $G = G' = G''$, u, v being the identity, and $t''_i = t_i$, we see that the various formal Lie groups $G^*((t_i))$ corresponding to different systems of uniformizing parameters, are isomorphic. We will therefore usually write from now on G^* to denote one of these groups.

Suppose H is a closed subgroup of dimension $m < n$ of the algebraic group G, and take a system (t_i) of uniformizing parameters of $\mathfrak{o}_K(e, G_0)$ such that, if u is the natural injection of H into G, the kernel of $\bar{u}$ is generated by the t_i of index $> m$, and the elements $\bar{u}(t_i)$ for $i \leqq m$ form a system of uniformizing parameters of $\mathfrak{o}_K(e, H_0)$. With that choice of uniformizing parameters, it is clear that the formal Lie group H^* will be the typical subgroup of G^* defined by the equations $x_i = 0$ for $i > m$.

PROPOSITION 18. *If u is a rational homomorphism of an algebraic group G into an algebraic group H, having kernel N, then, if $\boldsymbol{u}$ is the*

[11] From now on, we will assume (unless the contrary is explicitly stated) that whenever the field K is mentioned in connection with algebraic groups or rational homomorphisms, it is a common field of definition of all points, algebraic groups and rational homomorphisms concerned. We recall that no fields other than *perfect* fields are ever considered in this paper, which suppresses the distinction between field of definition and "field of quasi-definition" [2, p. 25].

corresponding homomorphism[12] *of* G^* *into* H^*, N^* *is the kernel of* $\boldsymbol{u}$, $\boldsymbol{u}(G^*) = (u(G))^*$ *and the rank of* $\boldsymbol{u}$ *is equal to the rank of* u.

Let K be a common field of definition of G, H and u. As u can be considered as composed of the natural injection of $u(G)$ into H and of a surjective homomorphism defined over K [16, p. 415, cor. 1 to th. 4], we can immediately restrict ourselves to the case in which u is *surjective*. If v is the natural injection of N into G, $\boldsymbol{v}$ the corresponding monomorphism of N^* into G^*, then from $u(v(N)) = \{e'\}$ follows $\boldsymbol{u}(\boldsymbol{v}(N^*)) = \boldsymbol{e}'$, in other words N^* is contained in the kernel N' of $\boldsymbol{u}$. Suppose $N' \neq N^*$; then, if $\dim G = n$, $\dim H = m$, we would have $\dim N' > n - m$ [2, prop. 5.2, p. 37]; by a change of variables, we may suppose N' is a typical subgroup of G^*, and then the definition of $\boldsymbol{u}$ shows that the image of $\mathfrak{O}_K(e', H_0)$ by $\bar{u}$ would be a ring of power series in less than m indeterminates; but this contradicts the fact that $\bar{u}$ is an *isomorphism* of $\mathfrak{O}_K(e', H_0)$ onto its image in $\mathfrak{O}_K(e, G_0)$. Then the dimensions of $\boldsymbol{u}(G^*)$ and $(u(G))^*$ are equal, and this ends the proof.

This result implies immediately that, for any algebraic subgroup F of G, $\boldsymbol{u}(F^*) = (u(F))^*$, and, for any algebraic subgroup $F' \subset u(G)$, $\boldsymbol{u}^{-1}(F'^*) = (u^{-1}(F'))^*$,[13] the latter relation being again due to the fact that the dimensions of both groups are the same, and that the first contains the second by definition.

If H_1, H_2 are two algebraic subgroups of G which *commute* (elementwise), it is immediately verified that H^*_1 and H^*_2 commute in G^*.

PROPOSITION 19. *If* N_1, N_2 *are two connected normal algebraic subgroups of* G, *then* $[N_1, N_2]^* = [N^*_1, N^*_2]$.

If u is the natural homomorphism of G onto $G/[N_1, N_2]$, $u(N_1)$ and $u(N_2)$ commute, hence $\boldsymbol{u}(N^*_1)$ and $\boldsymbol{u}(N^*_2)$ commute in $\boldsymbol{u}(G^*)$; as $[N_1, N_2]^*$ is the kernel of $\boldsymbol{u}$, we have $[N^*_1, N^*_2] \subset [N_1, N_2]^*$ by definition (no. 13). Let $\dim[N^*_1, N^*_2] = m$, $\dim[N_1, N_2] = q$. We can suppose the uniformizing parameters t_i $(1 \leqq i \leqq n)$ of $\mathfrak{o}_K(e, G_0)$ have been taken so that, if v is the natural injection of $[N_1, N_2]$ into G, the $s_i = \bar{v}(t_i)$ $(1 \leqq i \leqq q)$, constitute a system of uniformizing parameters of $\mathfrak{o}_K(e, [N_1, N_2]_0)$. We identify the t_i $(1 \leqq i \leqq n)$ with a system $\boldsymbol{x} = (x_1, \cdots, x_n)$ of n indeterminates, the s_i

[12] I will systematically use the convention by which a boldface letter represents the homomorphism of formal Lie groups corresponding to a rational homomorphism of algebraic groups, denoted by the corresponding italic letter.

[13] If U is the graph of u, which is an algebraic subgroup of $G \times G'$, $u^{-1}(F')$ is the projection on G of the subgroup $U \cap (G \times F')$ of $G \times G'$.

$(1 \leqq i \leqq q)$ with a system $\mathbf{y} = (y_1, \cdots, y_q)$ of q indeterminates. We know it is possible to "change variables" in G^* so as to transform $[N^*_1, N^*_2]$ into a typical subgroup: this means there is a system of n power series $x'_i = \psi_i(x_1, \cdots, x_n)$ with non vanishing jacobian such that $[N^*_1, N^*_2]$ is defined by the relations $x'_i = 0$ for $m+1 \leqq i \leqq n$.

Consider now r generic points n'_j (resp. n''_j) of N_1 (resp. N_2) over K, all these points being independent; let f (resp. g) be the natural injection of N_1 (resp. N_2) into G, and consider the rational map F of $N^r_1 \times N^r_2$ into G, defined by

$$F((n'_j), (n''_j)) = (f(n'_1))(g(n''_1))(f(n'_1))^{-1}(g(n''_1))^{-1} \cdots (f(n'_r))(g(n''_r))(f(n'_r))^{-1}(g(n''_r))^{-1};$$

by definition, one can write $F = v \circ F_1$, where F_1 is a rational map of $N^r_1 \times N^r_2$ into $[N_1, N_2]$. Take in $\mathfrak{o}_K(e, (N_1)_0)$ (resp. $\mathfrak{o}_K(e, (N_2)_0)$) a system $(w'_1, \cdots, w'_{d_1})$ (resp. $(w''_1, \cdots, w''_{d_2})$) of uniformizing parameters, and in $\mathfrak{o}_K((e, \cdots, e), (N^r_1 \times N^r_2)_0)$ the system of uniformizing parameters

$$w'_{11}, \cdots, w'_{1d_1}, w'_{21}, \cdots, w'_{2d_1}, \cdots, w'_{r1}, \cdots, w'_{rd_1},$$
$$w''_{11}, \cdots, w''_{1d_2}, \cdots, w''_{r1}, \cdots, w''_{rd_2}$$

which corresponds naturally to the w'_h and w''_h. If we identify these parameters with $2r$ systems $\mathbf{z}'_j = (z'_{jh})_{1 \leqq h \leqq d_1}$, $\mathbf{z}''_j = (z''_{jk})_{1 \leqq k \leqq d_2}$ of indeterminates $(1 \leqq j \leqq r)$, then $\bar{F}(t_i) = 0$ for $q+1 \leqq i \leqq n$, and $\bar{F}(t_i) = \bar{F}_1(s_i)$ for $1 \leqq i \leqq q$ is a power series $w_i((\mathbf{z}'_j), (\mathbf{z}''_j))$ in the z'_{jh} and z''_{jk}. But it follows from the definitions that this system of n power series is the one which we have agreed to denote by

$$\boldsymbol{f}(\mathbf{z}'_1)\boldsymbol{g}(\mathbf{z}''_1)\boldsymbol{f}(\mathbf{z}'_1)^{-1}\boldsymbol{g}(\mathbf{z}''_1)^{-1} \cdots \boldsymbol{f}(\mathbf{z}'_r)\boldsymbol{g}(\mathbf{z}''_r)\boldsymbol{f}(\mathbf{z}'_r)^{-1}\boldsymbol{g}(\mathbf{z}''_r)^{-1}$$

and from the definition of $[N^*_1, N^*_2]$ (no. 13) it follows that we have

$$\psi_i(w_1((\mathbf{z}'_j), (\mathbf{z}''_j)), \cdots, w_q((\mathbf{z}'_j), (\mathbf{z}''_j)), 0, \cdots, 0) = 0 \quad \text{for } m+1 \leqq i \leqq n. \tag{13}$$

Now it is known [2, pp. 34-35] that for r large enough, $F_1((n'_j), (n''_j))$ is a *generic point* of $[N_1, N_2]$. But then $\bar{F}_1$ is an *isomorphism* of $\mathfrak{D}_K(e, [N_1, N_2]_0)$ onto its image; and for $m+1 \leqq i \leqq n$, the left hand side of (13) is the image by $\bar{F}_1$ of the element of $\mathfrak{D}_K(e, [N_1, N_2]_0)$ identified to the power series $\psi_i(y_1, \cdots, y_q, 0, \cdots, 0)$. Therefore we have $\psi_i(y_1, \cdots, y_q, 0, \cdots, 0) = 0$ for $m+1 \leqq i \leqq n$; but by definition, this means that $v([N_1, N_2]^*) \subset [N^*_1, N^*_2]$, q. e. d.

COROLLARY. *In order that a connected algebraic group G be solvable* (resp. *nilpotent, commutative*), *it is necessary and sufficient that G^* be solvable* (resp. *nilpotent, commutative*).

15. Representable groups. By *general linear group* $GL(n)$, we will always mean the algebraic group consisting of all $n \times n$ invertible matrices with elements in the "universal domain" Ω; it is connected and defined over the prime field $\boldsymbol{F}_p$. If (t_{ij}) are the elements of a generic matrix $T \in GL(n)$ over $\boldsymbol{F}_p$, then we can take the $t_{ij} - \delta_{ij}$ as uniformizing parameters of $\mathfrak{o}_{\boldsymbol{F}_p}(e, GL(n))$, which we call the *coordinates* of T. The corresponding formal Lie group over $\boldsymbol{F}_p$ will be denoted $GL^*(n)$; its group law is defined by the power series $\phi_{ij}(\boldsymbol{x}, \boldsymbol{y})$ (with $\boldsymbol{x} = (x_{ij})$, $\boldsymbol{y} = (y_{ij})$, $1 \leqq i, j \leqq n$)

$$\delta_{ij} + \phi_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{n} (\delta_{ik} + x_{ik})(\delta_{kj} + y_{kj}). \tag{14}$$

A homomorphism $\boldsymbol{u}$ of a formal Lie group G (over a field K) into $GL^*(n)$ is therefore a system (u_{ij}) of n^2 power series (with coefficients in K) verifying the identities

$$\delta_{ij} + u_{ij}(\boldsymbol{xy}) = \sum_{k=1}^{n} (\delta_{ik} + u_{ik}(\boldsymbol{x}))(\delta_{kj} + u_{kj}(\boldsymbol{y})). \tag{15}$$

Suppose $G = G'^*$, where G' is an algebraic group, and let u be a rational homomorphism of G' into $GL(n)$. Then, if t is a generic point of G'_0 over K, $u(t)$ is a matrix $(\delta_{ij} + u_{ij}(t))$ where the $u_{ij}(t)$ are elements of the local ring $\mathfrak{o}_K(e, G'_0)$; taking uniformizing parameters and identifying $\mathfrak{D}_K(e, G'_0)$ with a ring of power series, the $u_{ij}(t)$ are identified to power series $\bar{u}_{ij}(\boldsymbol{x})$, and $\boldsymbol{u} = (\bar{u}_{ij}(\boldsymbol{x}))$ is the homomorphism of G into $GL^*(n)$, corresponding to u.

LEMMA 2. *Let $\boldsymbol{u}$ be a homomorphism of a formal Lie group G (over a field K) into $GL^*(n)$, and let H be an algebraic subgroup of $GL(n)$, defined over K. Suppose that H is the set of matrices of coordinates z_{ij} such that $P_\lambda(z_{11}, \cdots, z_{nn}) = 0$, where (P_λ) is a family of polynomials in $K[X_{11}, \cdots, X_{nn}]$. Then, in order that $\boldsymbol{u}(G) \subset H^*$, it is necessary and sufficient that the power series $P_\lambda(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x})) = 0$ for each λ.*

Let $\mathfrak{c}$ be the ideal generated by the P_λ, $\mathfrak{b} \supset \mathfrak{c}$ the ideal of H. We first observe that the condition is equivalent to saying that $P(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x})) = 0$ for every $P \in \mathfrak{b}$, since $P^m \in \mathfrak{c}$ for m large enough and the ring of power series has no divisors of 0. Next, if $\mathfrak{b}_0 \supset \mathfrak{b}$ is the ideal of the connected component H_0 of H, it is also equivalent to say that $P(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x}))$

$=0$ for every $P \in \mathfrak{b}_0$. Indeed, the polynomials $P \in \mathfrak{b}_0$ are characterized by the property that $PQ_0 \in \mathfrak{b}$, where Q_0 is a polynomial with nonvanishing constant term [6, p. 86, th. 2], and as the $u_{ij}(\boldsymbol{x})$ have no constant term, it is clear that the power series $Q_0(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x}))$ has a nonvanishing constant term, i.e. is invertible. We can therefore suppose H is *connected*.

The condition is necessary, for if $t = (\delta_{ij} + t_{ij})$ is a generic point of H over K, the t_{ij} belong to $\mathfrak{o}_K(e, H)$, and $P(t_{11}, \cdots, t_{nn}) = 0$ for any $P \in \mathfrak{b}$; let $(s_k)_{1 \leq k \leq m}$ be a system of uniformizing parameters of $\mathfrak{o}_K(e, H)$; then $\boldsymbol{u}$ is defined by a system of m power series $v_k(\boldsymbol{x})$, and $u_{ij}(\boldsymbol{x})$ is the series obtained by substituting $v_k(\boldsymbol{x})$ to s_k in the expression of t_{ij} as power series in the s_k, hence $P(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x})) = 0$.

To prove the converse, note that $K(t)$ is the field of fractions of the quotient ring $K[X_{11}, \cdots, X_{nn}]/\mathfrak{b} = \mathfrak{R}$; let $\bar{X}_{ij}$ be the class mod $\mathfrak{b}$ of X_{ij} in $\mathfrak{R}$. Then we can write $s_k = P_k(\bar{X}_{11}, \cdots, \bar{X}_{nn})/Q_k(\bar{X}_{11}, \cdots, \bar{X}_{nn})$, where P_k and Q_k are polynomials of $K[X_{11}, \cdots, X_{nn}]$ such that $P_k(0) = 0$, $Q_k(0) \neq 0$, the polynomials P_k, Q_k being only determined mod $\mathfrak{b}$. From the assumption on the $u_{ij}(\boldsymbol{x})$, the elements

$$v_k(\boldsymbol{x}) = P_k(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x}))/Q_k(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x}))$$

are well determined power series without constant terms. On the other hand, if w is the natural injection of H into $GL(n)$, there exists a system of uniformizing parameters $(t'_k)_{1 \leq k \leq n^2}$ of $\mathfrak{o}_K(e, GL(n))$, such that $s_k = \bar{w}(t'_k)$ for $1 \leqq k \leqq m$, and that $\bar{w}(t'_k) = 0$ for $k > m$, in other words the t'_k for $k > m$ are in the ideal generated by $\mathfrak{b}$ in the local ring $\mathfrak{o}_K(e, GL(n))$. The t'_k are power series in the coordinates of a generic matrix of $GL(n)$ (which form, as we have seen, a system of uniformizing parameters of $\mathfrak{o}_K(e, GL(n))$); substituting the $u_{ij}(\boldsymbol{x})$ to these coordinates in these power series, we obtain 0 for $k > m$ and $v_k(\boldsymbol{x})$ for $1 \leqq k \leqq m$, which shows that the $v_k(\boldsymbol{x})$ define a homomorphism of G into H^* such that $\boldsymbol{u} = \boldsymbol{w} \circ \boldsymbol{v}$.

We will say that a formal Lie group G is *representable* if it is *isogenous to a subgroup of a group* $GL^*(n)$. As we will chiefly be interested in properties invariant under isogeny, we will mainly consider subgroups of $GL^*(n)$.

PROPOSITION 20. *For any formal Lie group G, the center $Z(G)$ is such that $G/Z(G)$ is representable.*

PROPOSITION 21. *For any algebraic group G', $Z(G'^*) = (Z(G'))^*$.*

Both propositions will be proved by the method used by M. Rosenlicht to show that if G' is an algebraic group, $G'/Z(G')$ is isogenous to an algebraic

matrix group ([16, p. 432, th. 13]; see also [18]). Let $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{z} = (z_1, \cdots, z_n)$ be independent generic points of the formal Lie group G, L a perfect field containing the ring $K[[z_1, \cdots, z_n]]$, and $\mathfrak{o}$ the ring of power series $L[[x_1, \cdots, x_n]]$; if $\mathfrak{m}$ is the maximal ideal of $\mathfrak{o}$ (power series without constant terms) then, for any $h > 0$, $\mathfrak{m}^h$ is the ideal of power series having only terms of total degree $\geqq h$. If to each $f \in \mathfrak{m}$ we associate the power series $f_{\boldsymbol{z}}(\boldsymbol{x}) = f(\boldsymbol{z}^{-1}\boldsymbol{x}\boldsymbol{z})$, it is clear that $f \to f_{\boldsymbol{z}}$ maps $\mathfrak{m}^h$ into $\mathfrak{m}^h$, hence defines, for each h, a homomorphism $j_{\boldsymbol{z}}$ of $\mathfrak{m}/\mathfrak{m}^h$ into itself, such that $j_{\boldsymbol{z}} j_{\boldsymbol{z}^{-1}} = 1$; $j_{\boldsymbol{z}}$ is therefore an automorphism of the L-vector space $\mathfrak{m}/\mathfrak{m}^h$. The classes $\bar{\boldsymbol{x}}^{\alpha}$ of the monomials $\boldsymbol{x}^{\alpha}$ such that $|\alpha| < h$ form a basis for $\mathfrak{m}/\mathfrak{m}^h$, hence we may write $j_{\boldsymbol{z}}(\bar{\boldsymbol{x}}^{\alpha}) = \sum_{|\beta|<h} c_{\alpha\beta}(\boldsymbol{z})\bar{\boldsymbol{x}}^{\beta}$ for $|\alpha| < h$. It is then clear that $\boldsymbol{z} \to (c_{\alpha\beta}(\boldsymbol{z}) - \delta_{\alpha\beta})$ is a *homomorphism* $\boldsymbol{u}_h$ of the group G into $GL^*(n_h)$, where $n_h = \dim \mathfrak{m}/\mathfrak{m}^h$. Let Z_h be the kernel of $\boldsymbol{u}_h$; then we have obviously $Z_{h+1} \subset Z_h$, and therefore there is a smallest h such that $Z_k = Z_h$ for all $k \geqq h$. If $\boldsymbol{v}$ is the natural injection of Z_h into G, $\mathbf{y}$ a generic point of Z_h, we have therefore $f - f_{v(y)} \in \mathfrak{m}^k$ for *any* $k \geqq h$, i.e. $f = f_{v(y)}$ for all $f \in \mathfrak{m}$, and this shows by definition that $Z_h \subset Z(G)$. On the other hand, if $\boldsymbol{w}$ is the natural injection of $Z(G)$ into G, and $\boldsymbol{t}$ a generic point of $Z(G)$, we have $f_{w(t)} = f$ for all $f \in \mathfrak{m}$, hence $Z(G) \subset Z_k$ for any k, and therefore $Z(G) = Z_h$, and $G/Z(G)$ is isogenous to a subgroup of $GL^*(n_h)$ by Proposition 11, which proves Proposition 20.

Suppose now $G = G'^*$; we define in a similar way (M. Rosenlicht, *loc. cit.*) for any $h \geqq 1$ a rational homomorphism $u_h : s \to (c'_{\alpha\beta}(s) - \delta_{\alpha\beta})$ of G' into $GL(n_h)$; and if we take a system of uniformizing parameters in $\mathfrak{o}_K(e, G'_0)$ and identify it with $\boldsymbol{z}$, then it follows from the definitions that $c_{\alpha\beta}(\boldsymbol{z})$ is the power series identified to $c'_{\alpha\beta}(s)$ (when s is a generic point of G'_0 over K). In other words, $\boldsymbol{u}_h$ is the homomorphism corresponding to u_h; hence, by Proposition 18, if Z'_h is the kernel of u_h, $Z_h = Z'^*_h$. But (*loc. cit.*) Z'_h is equal to the center $Z(G')$ for sufficiently large h; hence this proves Proposition 21.

16. Algebraic hull of a subgroup of $GL^*(n)$. In what follows, an *algebraic set* in $GL(n)$ will have the same meaning as in [2]. If M is an algebraic set in $GL(n)$, defined over K, $\mathfrak{b}$ its ideal in $K[X_{11}, \cdots, X_{nn}]$, G a formal Lie group over K and $\boldsymbol{u}$ a homomorphism of G into $GL^*(n)$, we say that $\boldsymbol{u}(G)$ is *contained in* M if, for any polynomial $P \in \mathfrak{b}$, the power series $P(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x})) = 0$. It is clear that if H is a subgroup of G (as defined in no. 5), then $\boldsymbol{u}(H) \subset M$. The same argument as in Lemma 2

shows that in order that $\boldsymbol{u}(G) \subset M$ it is sufficient that $P_\lambda(u_{ij}(\boldsymbol{x})) = 0$ for a family of polynomials $P_\lambda \in K[X_{11}, \cdots, X_{nn}]$ such that M is the set of matrices for which all P_λ are 0.

PROPOSITION 22. *For any homomorphism $\boldsymbol{u}$ of a formal Lie group G over K into $GL^*(n)$, the smallest algebraic set $\mathcal{A}(G, \boldsymbol{u})$ of $GL(n)$ containing $\boldsymbol{u}(G)$ is a connected subgroup of $GL(n)$, defined over K, and $\boldsymbol{u}(G)$ is contained in the formal Lie group $(\mathcal{A}(G, \boldsymbol{u}))^*$.*

Let $\mathfrak{b}$ be the ideal of $\mathcal{A}(G, \boldsymbol{u})$ in $K[X_{11}, \cdots, X_{nn}]$; we prove that if s, t are in $\mathcal{A}(G, \boldsymbol{u})$ then so is st, which will prove $\mathcal{A}(G, \boldsymbol{u})$ is a group [6, p. 82, prop. 2]. For any polynomial $P \in K[X_{11}, \cdots, X_{nn}]$, and any $s \in GL(n)$, let ${}_sP$ be the polynomial with coefficients in $K[s]$ obtained by substituting in P to the X_{ij} the coordinates of the matrix, product of s and of the matrix $(\delta_{ij} + X_{ij})$. We can then write

$$(16) \qquad {}_sP(u_{11}(\boldsymbol{y}), \cdots, u_{nn}(\boldsymbol{y})) = \sum_\alpha Q_\alpha(s_{11}, \cdots, s_{nn})\boldsymbol{y}^\alpha$$

where the Q_α are *polynomials* of $K[X_{11}, \cdots, X_{nn}]$ (since a term in $\mathbf{y}^\alpha$ can only come from terms of degree $\leqq |\alpha|$ in the $u_{ij}(\mathbf{y})$), and the s_{ij} the coordinates of s. Suppose $P \in \mathfrak{b}$; then we have

$$P(u_{11}(\boldsymbol{xy}), \cdots, u_{nn}(\boldsymbol{xy})) = 0$$

which, as $\boldsymbol{u}$ is a homomorphism, is equivalent to

$$\sum_\alpha Q_\alpha(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x}))\boldsymbol{y}^\alpha = 0$$

and implies therefore $Q_\alpha(u_{11}(\boldsymbol{x}), \cdots, u_{nn}(\boldsymbol{x})) = 0$ for each α. But by definition, this means that $Q_\alpha \in \mathfrak{b}$ for each α; hence, if $s \in \mathcal{A}(G, \boldsymbol{u})$, we have $Q_\alpha(s_{11}, \cdots, s_{nn}) = 0$ for each α, and therefore, by (16), ${}_sP \in \mathfrak{b}^{K(s)}$ (ideal generated by $\mathfrak{b}$ in $K(s)[X_{11}, \cdots, X_{nn}]$); hence $P(st) = 0$ for any $t \in \mathcal{A}(G, \boldsymbol{u})$, which proves our assertion. The argument in Lemma 2 shows moreover that $\mathcal{A}(G, \boldsymbol{u})$ is connected, and it also follows from Lemma 2 that $\boldsymbol{u}(G) \subset (\mathcal{A}(G, \boldsymbol{u}))^*$.

We shall simply write $\mathcal{A}(\boldsymbol{u}(G))$ instead of $\mathcal{A}(G, \boldsymbol{u})$, $\mathcal{A}^*(\boldsymbol{u}(G))$ instead of $(\mathcal{A}(G, \boldsymbol{u}))^*$; when $\boldsymbol{u}$ is a monomorphism (i.e. $(G, \boldsymbol{u})$ a subgroup of $GL^*(n)$) we will simply write $\mathcal{A}(G)$ and $\mathcal{A}^*(G)$. We say that $\mathcal{A}(G)$ is the *algebraic hull*[14] of $\boldsymbol{u}(G)$ in $GL(n)$. It is clear that if G' is a connected algebraic subgroup of $GL(n)$, $\mathcal{A}(G'^*) = G'$ and $\mathcal{A}^*(G'^*) = G'^*$.

[14] Here is an example of a formal Lie group $G \subset GL^*(2)$ which is distinct from its

Note that Proposition 22 is also valid when $GL(n)$ is replaced by an algebraic group G' which is an *open subset of an affine space*, the origin of the space being the neutral element of G'; the only modification in the proof is that the Q_α are replaced by rational functions R_α/S_α with $S_\alpha(0) \neq 0$. We shall make use of that remark when G' is a *product* of several groups $GL(n_k)$.

PROPOSITION 23. *If G is a formal Lie subgroup of $GL^*(n)$ and N a normal subgroup in G, then $\mathcal{A}(N)$ is a normal subgroup in $\mathcal{A}(G)$.*

Let $\mathfrak{b}$ be the ideal of $\mathcal{A}(N)$ in $K[X_{11}, \cdots, X_{nn}]$; we can suppose that the subgroup N is typical, and then, if $\boldsymbol{x}$ is a generic point of N, $\boldsymbol{z}$ a generic point of G, $\boldsymbol{u}$ the natural injection of G into $GL^*(n)$, we have $P(\boldsymbol{u}(\boldsymbol{z})\boldsymbol{u}(\boldsymbol{x})(\boldsymbol{u}(\boldsymbol{z}))^{-1}) = 0$ for any $P \in \mathfrak{b}$. For any $P \in K[X_{11}, \cdots, X_{nn}]$ and any $s \in GL(n)$, let ${}^sP$ be the polynomial with coefficients in $K(s)$ obtained by substituting to the X_{ij} (coordinates of the matrix $X = (\delta_{ij} + X_{ij})$) the coordinates of the matrix sXs^{-1}; we have then ${}^sP(\boldsymbol{u}(\boldsymbol{x})) = P(s\boldsymbol{u}(\boldsymbol{x})s^{-1}) = \sum_\alpha R_\alpha(s_{11}, \cdots, s_{nn})\boldsymbol{x}^\alpha$, where the R_α are rational functions defined at the origin. If $P \in \mathfrak{b}$, we have then as in Proposition 22, $R_\alpha(u_{11}(\boldsymbol{z}), \cdots, u_{nn}(\boldsymbol{z})) = 0$, in other words, the numerators of the R_α belong to the ideal $\mathfrak{c}$ of $\mathcal{A}(G)$. This proves that if $s \in \mathcal{A}(G)$, ${}^sP$ belongs to $\mathfrak{b}^{K(s)}$, hence that $sts^{-1} \in \mathcal{A}(N)$ for any $t \in \mathcal{A}(N)$.

PROPOSITION 24. *Let G', G'' be two connected algebraic groups which are open subsets of affine spaces, and f an everywhere defined rational map of G' into G''; suppose $\boldsymbol{u}$ is a homomorphism of a formal Lie group G into G'^*, M an algebraic subset of G'', and that for any polynomial P of the ideal $\mathfrak{b}$ of M, $P(f(\boldsymbol{u}(\boldsymbol{x}))) = 0$. Then $\mathcal{A}(\boldsymbol{u}(G)) \subset f^{-1}(M)$.*

For any generic point $s \in G'$, and any polynomial $P \in \mathfrak{b}$, we have $P(f(s)) = Q(s)/R(s)$ with Q, R polynomials in $K[s]$, and $R(0) \neq 0$, and $f^{-1}(M)$ is the set of matrices for which all the polynomials Q vanish. The remark preceding Proposition 22 then shows that $\boldsymbol{u}(G) \subset f^{-1}(M)$, hence the result.

PROPOSITION 25. a) *If G is a subgroup of $GL^*(n)$, G' a subgroup*

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algebraic hull $\mathcal{A}(G)$. We take G as the image of the multiplicative group I_0 by the homomorphism

$$x \to \begin{pmatrix} 1+x & 0 \\ 0 & (1+x)^\zeta \end{pmatrix}$$

where ζ is a *p-adic integer which is not a rational number* [9, p. 241]. G has therefore dimension 1, whereas from the characterization of the tori in $GL(n)$ [2, §7, pp. 42-46] and the choice of ζ, it follows at once that $\mathcal{A}(G)$ has dimension 2.

of $GL^*(n')$, *then* $\mathcal{A}(G\times G')=\mathcal{A}(G)\times\mathcal{A}(G')$.

b) *If* M,N *are two subgroups of* $GL^*(n)$, *and if* M *centralizes* N, *then* $\mathcal{A}(M)$ *centralizes* $\mathcal{A}(N)$.

c) *If* M,N *are two normal subgroups of a subgroup of* $GL^*(n)$, *then* $[\mathcal{A}(M),\mathcal{A}(N)]=\mathcal{A}([M,N])$.

d) *For any subgroup* G *of* $GL^*(n)$, $\mathcal{A}(D^k(G))=D^k(\mathcal{A}(G))$ *and* $\mathcal{A}(C^k(G))=C^k(\mathcal{A}(G))$.

e) *In order that a subgroup* G *of* $GL^*(n)$ *be commutative* (resp. *solvable, nilpotent*), *it is necessary and sufficient that* $\mathcal{A}(G)$ *be commutative* (resp. *solvable, nilpotent*).

f) *Let* G *be a subgroup of* $GL^*(n)$, N *a normal subgroup of* G, M *a subgroup of* G *containing* N. *Then if* M/N *is commutative* (resp. *solvable, nilpotent*), $\mathcal{A}(M)/\mathcal{A}(N)$ *is commutative* (resp. *solvable, nilpotent*).

These results are the counterparts of (3.2), (3.4), (4.5), (4.6), (4.7) and (4.8) in [2, pp. 33-35]; it will be observed that the proofs of these last results are purely formal consequences of the propositions corresponding to Propositions 22, 23 and 24 above, and of (4.3) in [2, p. 34]; the only point which needs some comment is the inclusion $[M,N]\subset[\mathcal{A}(M),\mathcal{A}(N)]^*$ (which is obvious in the context of [2]); this follows here from Proposition 19, which yields $[M,N]\subset[\mathcal{A}^*(M),\mathcal{A}^*(N)]=[\mathcal{A}(M),\mathcal{A}(N)]^*$, using Proposition 17 and Proposition 23. We omit the rest of the proofs.

THEOREM 4. *For any subgroup* G *of* $GL^*(n)$, $D(G)=\mathcal{A}^*(D(G))=(D(\mathcal{A}(G)))^*$, *and any normal subgroup of* G *is also a normal subgroup of* $\mathcal{A}^*(G)$.

This corresponds to Theorems 13 and 15 of [6, pp. 173 and 177] (and implies them when the characteristic is 0[10]); in non rigorous language, we can say that *the group of commutators of any representable formal Lie group "is algebraic", and is "equal" to the commutator subgroup of the algebraic hull of* G.

The proof is an adaptation of Chevalley's proof of his Theorem 13 (*loc. cit.*). We start with the group law of $GL^*(n)$ defined by (14). By Proposition 4, for any normal subgroup N of G, it is possible to "change variables" in $GL^*(n)$, in other words to find n^2 power series $x'_k=f_k(x_{11},\cdots,x_{nn})$ with nonvanishing jacobian, such that the subgroups $\mathcal{A}^*(G)$, G and N of $GL^*(n)$ be "defined" respectively by the equations $x'_k=0$ for $k>m$, $x'_k=0$ for $k>q$, $x'_k=0$ for $k>r$ $(r\leqq q\leqq m)$. Let

$x_{ij} = g_{ij}(x'_1, \cdots, x'_{n^2})$ be the "inverse" system of power series. Let $\mathbf{y} = (y_1, \cdots, y_r)$ be a generic point of N; denote by $\boldsymbol{g}(\mathbf{y})$ the matrix whose coordinates are the g_{ij} in which x'_k has been replaced by y_k for $k \leqq r$, by 0 for $k > r$, and consider, for a matrix s of coordinates s_{ij}, the power series $f_k(s\boldsymbol{g}(\mathbf{y})s^{-1})$ obtained by replacing in f_k the x_{ij} by the coordinates of the matrix $s\boldsymbol{g}(\mathbf{y})s^{-1}$; it is clear that we have $f_k(s\boldsymbol{g}(\mathbf{y})s^{-1}) = \sum_{\alpha} R_{k\alpha}(s_{11}, \cdots, s_{nn})\mathbf{y}^{\alpha}$, where the $R_{k\alpha} = P_{k\alpha}/Q_{k\alpha}$ are *rational* functions of $K(X_{11}, \cdots, X_{nn})$ such that $Q_{k\alpha}(0) \neq 0$. Now let $\mathbf{z} = (z_1, \cdots, z_q)$ be a generic point of G, $\boldsymbol{g}(\mathbf{z})$ the matrix whose coordinates are the g_{ij} in which x'_k is replaced by z_k for $k \leqq q$, by 0 for $k > q$. From the assumption that N is normal in G, it follows that for $k > r$, the power series $f_k(\boldsymbol{g}(\mathbf{z})\boldsymbol{g}(\mathbf{y})\boldsymbol{g}(\mathbf{z})^{-1})$ are 0, in other words that $R_{k\alpha}(\boldsymbol{g}(\mathbf{z})) = 0$ for all $k > r$ and all α. But by definition, this implies that $R_{k\alpha}(s_{11}, \cdots, s_{nn}) = 0$ if $s \in \mathfrak{A}(G)$; if $\mathbf{t} = (t_1, \cdots, t_m)$ is a generic point of $\mathfrak{A}^*(G)$, $\boldsymbol{g}(\mathbf{t})$ the matrix whose coordinates are the g_{ij} in which x'_k is replaced by t_k for $k \leqq m$, by 0 for $k > m$, we have therefore $R_{k\alpha}(\boldsymbol{g}(\mathbf{t})) = 0$ for all $k > r$ and all α, hence $f_k(\boldsymbol{g}(\mathbf{t})\boldsymbol{g}(\mathbf{y})\boldsymbol{g}(\mathbf{t})^{-1}) = 0$ for $k > r$, which means of course that N is a *normal* subgroup of $\mathfrak{A}^*(G)$.

Take now $N = D(G)$, which is therefore a normal subgroup of $\mathfrak{A}^*(G)$. With the same conventions as above, consider the power series

$$f_k(s\boldsymbol{g}(\mathbf{z})s^{-1}\boldsymbol{g}(\mathbf{z})^{-1}) = \sum_{\alpha} S_{k\alpha}(s_{11}, \cdots, s_{nn})\mathbf{z}^{\alpha},$$

where the $S_{k\alpha}$ are again *rational* functions. By assumption,

$$f_k(\boldsymbol{g}(\mathbf{z}')\boldsymbol{g}(\mathbf{z})\boldsymbol{g}(\mathbf{z}')^{-1}\boldsymbol{g}(\mathbf{z})^{-1}) = 0 \text{ for } k > r,$$

when $\mathbf{z}' = (z'_1, \cdots, z'_q)$ is a second generic point of G; this means that $S_{k\alpha}(\boldsymbol{g}(\mathbf{z}')) = 0$ for all $k > r$ and all α, and we conclude as above that $f_k(s\boldsymbol{g}(\mathbf{z})s^{-1}\boldsymbol{g}(\mathbf{z})^{-1}) = 0$ for $k > r$ and $s \in \mathfrak{A}(G)$, hence

$$f_k(\boldsymbol{g}(\mathbf{t})\boldsymbol{g}(\mathbf{z})\boldsymbol{g}(\mathbf{t})^{-1}\boldsymbol{g}(\mathbf{z})^{-1}) = 0 \text{ for } k > r.$$

Consider finally the power series $f_k(\boldsymbol{g}(\mathbf{t})s\boldsymbol{g}(\mathbf{t})^{-1}s^{-1}) = \sum_{\alpha} T_{k\alpha}(s_{11}, \cdots, s_{nn})\mathbf{t}^{\alpha}$ with rational $T_{k\alpha}$; the same argument proves that

$$f_k(\boldsymbol{g}(\mathbf{t})\boldsymbol{g}(\mathbf{t}')\boldsymbol{g}(\mathbf{t})^{-1}\boldsymbol{g}(\mathbf{t}')^{-1}) = 0 \text{ for } k > r,$$

when $\mathbf{t}' = (t'_1, \cdots, t'_m)$ is a second generic point of $\mathfrak{A}^*(G)$. As we know that $D(G)$ is a normal subgroup of $\mathfrak{A}^*(G)$, we conclude that $D(\mathfrak{A}^*(G)) \subset D(G)$; but clearly $D(G) \subset D(\mathfrak{A}^*(G))$, hence $D(G) = D(\mathfrak{A}^*(G))$. Now from Proposition 19 it follows that $D(\mathfrak{A}^*(G)) = (D(\mathfrak{A}(G)))^*$, and from Proposition 25 d), $D(\mathfrak{A}(G)) = \mathfrak{A}(D(G))$, whence $D(G) = \mathfrak{A}^*(D(G))$.

17. Normalizers and centralizers.

PROPOSITION 26. *Let G be a formal Lie group over K, M, N two subgroups of G. Then there is a largest subgroup P of M such that, if $\boldsymbol{u}$ is the natural injection of P into G, $\boldsymbol{i}_{\boldsymbol{u}(\boldsymbol{z})}(N_{(L)}) \subset N_{(L)}$, where $\boldsymbol{z}$ is a generic point of P and L a perfect field containing $K[[\boldsymbol{z}]]$.*

Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a generic point of G, and suppose N is a typical subgroup defined by $x_i = 0$ for $i > q$. Let $\boldsymbol{y} = (y_1, \cdots, y_m)$ be a generic point of M, E a perfect field containing $K[[\boldsymbol{y}]]$, $\mathfrak{o}$ the ring of power series $E[[\boldsymbol{x}]]$, $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. As in Proposition 20, we define an automorphism $j_{\boldsymbol{y}}$ of the E-vector space $\mathfrak{m}/\mathfrak{m}^h$ by associating to each $f \in \mathfrak{m}$ the power series $f_{\boldsymbol{y}}(\boldsymbol{x}) = f(\boldsymbol{v}(\boldsymbol{y})^{-1}\boldsymbol{x}\boldsymbol{v}(\boldsymbol{y}))$ ($\boldsymbol{v}$ natural mapping of M into G), and taking classes mod $\mathfrak{m}^h$; and with the same notations as in Proposition 20, we may write $j_{\boldsymbol{y}}(\bar{\boldsymbol{x}}^{\alpha}) = \sum_{|\beta|<h} c_{\alpha\beta}(\boldsymbol{y})\boldsymbol{x}^{\beta}$, thus obtaining a homomorphism $\boldsymbol{w}_h$: $\boldsymbol{y} \to (c_{\alpha\beta}(\boldsymbol{y}) - \delta_{\alpha\beta})$ of M into $GL^*(n_h)$. Let J be the interval $1 \leqq i \leqq q$, and let V_h be the subspace of $\mathfrak{m}/\mathfrak{m}^h$ having as a basis the $\boldsymbol{x}^{\alpha}$ such that $\alpha \in \boldsymbol{N}^J$. It is clear that the set of matrices $s \in GL(n_h)$ such that $s(V_h) \subset V_h$ is an algebraic subgroup S_h of $GL(n_h)$. Let Q_h be the connected component of the identity in the algebraic group $S_h \cap \mathcal{A}(\boldsymbol{w}_h(M))$; in the formal Lie group $\mathcal{A}^*(\boldsymbol{w}_h(M))$, $\boldsymbol{w}_h(M)$ is a normal subgroup by Theorem 4, hence the g.l.b. $Q^*_h \wedge \boldsymbol{w}_h(M)$ is defined (Proposition 9), and therefore we can also define the inverse image $P_h = \boldsymbol{w}^{-1}{}_h(Q^*_h)$ in M (see remark at the end of no. 8). These definitions imply that P_h is the *largest* subgroup of M such that $j_{\boldsymbol{t}(t)}(V_h) \subset V_h$, if $\boldsymbol{t}$ is a generic point of P_h and $\boldsymbol{f}$ the natural mapping of P_h into M. Furthermore, from that definition it follows that $P_{h+1} \subset P_h$, hence there is an integer r such that $P_h = P_r$ for $h \geqq r$, and $P = P_r$ obviously satisfies the requirements of Proposition 26.

When $M = G$, the subgroup P obviously contains N, and is the largest subgroup of G such that N is a normal subgroup of P; we say, as usual, that P is the *normalizer* of N in G. Proposition 26 then expresses the fact that for any subgroup M of G, the g.l.b. $P \wedge M$ *exists*. This enables us to prove the result announced in no. 5, namely:

THEOREM 5. *The subgroups of a formal Lie group G form a complete lattice.*

Owing to the chain conditions, all we need to do is to prove that $H_1 \wedge H_2$ exists for any pair of subgroups of G. Consider the normalizer N_2 of H_2 in G; any subgroup contained in H_1 and H_2 is contained in H_1

and N_2, hence in $H_1 \wedge N_2$, which is defined as we have seen above. Moreover, in the group N_2, H_2 is a normal subgroup, hence $H_2 \wedge (H_1 \wedge N_2)$ is defined (Proposition 9), and is obviously the g.l.b. of H_1 and H_2.

COROLLARY 1. *Let $\boldsymbol{u}$ be a surjective homomorphism of G onto a formal Lie group G', H_1, H_2 two subgroups of G, H'_1, H'_2 two subgroups of G'; then*

$$\boldsymbol{u}(H_1 \vee H_2) = \boldsymbol{u}(H_1) \vee \boldsymbol{u}(H_2) \tag{17}$$

$$\begin{aligned} \boldsymbol{u}^{-1}(H'_1 \vee H'_2) &= \boldsymbol{u}^{-1}(H'_1) \vee \boldsymbol{u}^{-1}(H'_2), \\ \boldsymbol{u}^{-1}(H'_1 \wedge H'_2) &= \boldsymbol{u}^{-1}(H'_1) \wedge \boldsymbol{u}^{-1}(H'_2). \end{aligned} \tag{18}$$

Formulae (18) follow at once from the fact that $H' \to \boldsymbol{u}^{-1}(H')$ is a one-to-one inclusion preserving correspondence between subgroups of G' and subgroups of G containing the kernel N of $\boldsymbol{u}$ (Proposition 11); the same argument proves (17), when we observe that

$$H_1 \vee H_2 \vee N = (H_1 \vee N) \vee (H_2 \vee N),$$

and that $\boldsymbol{u}(H) = \boldsymbol{u}(H \vee N)$ for any subgroup H of G.

COROLLARY 2. *Let H_1, H_2 be two algebraic subgroups of $GL(n)$, $H_1 \vee H_2$* (resp. $H_1 \wedge H_2$) *the smallest algebraic subgroup of $GL(n)$ containing H_1 and H_2* (resp. *the largest algebraic subgroup of $GL(n)$ contained in H_1 and H_2*). *Then $(H_1 \wedge H_2)^* = H^*_1 \wedge H^*_2$, and, if H_1 and H_2 are connected, $(H_1 \vee H_2)^* = H^*_1 \vee H^*_2$.*

Note that $H_1 \wedge H_2$ is the set theoretic intersection $H_1 \cap H_2$, and is defined over K if both H_1 and H_2 are. We have obviously $(H_1 \wedge H_2)^* \subset H^*_1 \wedge H^*_2$; on the other hand $\mathfrak{A}(H^*_1 \wedge H^*_2)$ is contained in both $\mathfrak{A}(H^*_1)$ and $\mathfrak{A}(H^*_2)$, which are respectively the connected components of e in H_1 and H_2; hence $H^*_1 \wedge H^*_2$ is contained in the connected component of $H_1 \wedge H_2$, which proves that $H^*_1 \wedge H^*_2 \subset (H_1 \wedge H_2)^*$ by definition. On the other hand, we have obviously $H^*_1 \vee H^*_2 \subset (H_1 \vee H_2)^*$. To prove the converse inclusion, we remark that if x_i $(1 \leqq i \leqq m)$ are independent generic points of H_1, y_i $(1 \leqq i \leqq m)$ independent generic points of H_2, over a common field of definition K, then, if m is large enough, $x_1 y_1 x_2 y_2 \cdots x_m y_m$ is a generic point of $H_1 \vee H_2$ over K; this is easily proved as the Lemmas 3.1 and 4.3 of [2]. The rest of the argument is then practically identical to that in Proposition 19, and we accordingly suppress it.

PROPOSITION 27. *For any subgroup H of a formal Lie group G, there is a largest subgroup H' which commutes with H* (no. 11). *That subgroup*

H' (called the *centralizer* of H in G) *is a normal subgroup of the normalizer of* H.

With the notations of Proposition 26 (H replacing N and G replacing M), we consider the algebraic subgroup T_h of $GL(n_h)$ consisting of the matrices s such that the restriction of s to V_h is the identity, and the inverse image $w^{-1}{}_h(T^*{}_h)$ in G. These subgroups form a decreasing sequence of subgroups of G, and their intersection is the group H'. If N is the normalizer of H in G, it is clear that $H' \subset N$. To prove H' is normal in N, we may suppose $N = G$. Then, if z is a generic point of G, $i_z(H'_{(L)})$ commutes with $i_z(H_{(L)})$ (L perfect field containing $K[[z]]$), and as the latter is equal to $H_{(L)}$ we have by definition $i_z(H'_{(L)}) \subset H'_{(L)}$.

For representable groups, we have additional results. For any $s \in GL(n)$, let a_s be the rational map $t \to sts^{-1}$ of $GL(n)$ onto itself, defined over the field $\boldsymbol{F}_p(s)$; it gives rise to an automorphism $\boldsymbol{a}_s$ of the formal Lie group $(GL^*(n))_{(\boldsymbol{F}_p(s))}$.

PROPOSITION 28. *Let G be a subgroup (over a field K) of $GL^*(n)$; the set of elements $s \in GL(n)$ such that $a_s(G_{(K(s))}) \subset G_{(K(s))}$ (resp. such that the restriction of a_s to $G_{(K(s))}$ is the identity mapping) is an algebraic subgroup $\mathfrak{N}(G)$ (resp. $\mathfrak{Z}(G)$) of $GL(n)$, defined over K and such that $\mathfrak{N}^*(G) = (\mathfrak{N}(G))^*$ (resp. $\mathfrak{Z}^*(G) = (\mathfrak{Z}(G))^*$) is the normalizer (resp. centralizer) of G in $GL^*(n)$.*

First, for reasons of dimension, if $a_s(G_{(K(s))}) \subset G_{(K(s))}$, both groups are equal, from which it follows that if $s \in \mathfrak{N}(G)$, then $s^{-1} \in \mathfrak{N}(G)$; if $t \in \mathfrak{N}(G)$ and $L = K(s,t)$, we have $a_s(G_{(L)}) = G_{(L)}$ and $a_t(G_{(L)}) = G_{(L)}$; as $a_{st} = a_s \circ a_t$, we have $a_{st}(G_{(L)}) = G_{(L)}$, hence $st \in \mathfrak{N}(G)$, and $\mathfrak{N}(G)$ is therefore a group. On the other hand, taking G to be a typical subgroup of $GL^*(n)$ by a suitable "change of variables," the same argument as in Theorem 4 proves that the elements $s \in \mathfrak{N}(G)$ are characterized as the matrices satisfying a family of equations $R_{k\alpha}(s_{11}, \cdots, s_{nn}) = 0$, where the $R_{k\alpha}$ are rational functions of $K(X_{11}, \cdots, X_{nn})$. Lastly, if H is a subgroup of $GL^*(n)$, z a generic point of H and u the natural mapping of H into $GL^*(n)$, then the condition $i_{u(z)}(G_{(E)}) \subset G_{(E)}$ (for a perfect field E containing $K[[z]]$) is equivalent to the relations $R_{k\alpha}(u_{11}(z), \cdots, u_{nn}(z)) = 0$ for all indices, and the fact that $\mathfrak{N}^*(G)$ is the normalizer of G in $GL^*(n)$ then follows from Lemma 2. A similar proof applies to the centralizer.

COROLLARY 1. *For any normal subgroup N of G, $\mathfrak{A}(G)$ is contained in $\mathfrak{N}(N)$.*

This follows from the first part of the proof of Theorem 4.

COROLLARY 2. *$\mathfrak{N}(G)$ is contained in the normalizer of the algebraic hull $\mathfrak{A}(G)$ in $GL(n)$, and $\mathfrak{Z}(G)$ is equal to the centralizer of $\mathfrak{A}(G)$ in $GL(n)$.*

The first statement follows from Proposition 23, applied to the normal subgroup G of $\mathfrak{N}^*(G)$; the second follows likewise from Proposition 25 b) and from the fact that $\mathfrak{Z}(G)$ is a decreasing function of G.

18. Solvable representable groups. Let G be a solvable subgroup of $GL^*(n)$; then by Proposition 25 e), $\mathfrak{A}(G)$ is solvable. We denote by $\mathfrak{T}(n)$ the subgroup of *triangular* matrices $s=(s_{ij})$ (with $s_{ij}=0$ for $i>j$), by $\mathfrak{U}(n)$ the normal subgroup of $\mathfrak{T}(n)$ consisting of *unipotent* triangular matrices (i.e. with diagonal elements 1). Applying the preceding remark and the Lie-Kolchin theorem [2, th. 10.4], we have the corresponding result for representable groups:

PROPOSITION 29. *If G is a solvable subgroup of $GL^*(n)$ defined over an algebraically closed field K, there is an element $s\in GL(n)$, rational over K, such that $\mathfrak{a}_s(G)\subset\mathfrak{T}^*(n)$.*

In other words, if $\mathfrak{u}$ is the natural mapping of G into $GL^*(n)$, and $\mathfrak{v}=\mathfrak{a}_s\circ\mathfrak{u}$, then the matrix $1+\mathfrak{v}(\mathfrak{z})=(v_{ij}(\mathfrak{z})+\delta_{ij})$ is triangular ($\mathfrak{z}$ being a generic point of G).

PROPOSITION 30. *For any solvable formal Lie group G, the derived group $D(G)$ is nilpotent.*

This follows immediately from Proposition 29 if G is a subgroup of $GL^*(n)$, for we can then suppose that $G\subset\mathfrak{T}^*(n)$, and $D(G)\subset D(\mathfrak{T}^*(n))\subset\mathfrak{U}^*(n)$; as $\mathfrak{U}(n)$ is a nilpotent algebraic group, $\mathfrak{U}^*(n)$ is nilpotent by the corollary to Proposition 19. In general, if G is solvable, $G/Z(G)$ is solvable and representable, hence $D(G/Z(G))$ is nilpotent; but $D(G/Z(G))$ is isogenous to $D(G)/(Z(G)\wedge D(G))$ by Propositions 16 and 11, hence a fortiori $D(G)/Z(D(G))$, which is isogenous to a quotient group of $D(G)/(Z(G)\wedge D(G))$, is nilpotent. But this implies that $C^k(D(G))\subset Z(D(G))$ for some k (corollary to Proposition 16); by definition, we have $C^{k+1}(D(G))=\mathfrak{e}$, hence $D(G)$ is nilpotent.

We next prove the result corresponding to A. Borel's conjugation theorem for maximal solvable subgroups:

PROPOSITION 31. *Let G be a subgroup of $GL^*(n)$, over an algebraically closed field K. Then:*

a) *If S_1, S_2 are any two maximal solvable subgroups of G, there is an $s \in \mathfrak{A}(G)$, rational over K, such that $\mathfrak{a}_s(S_1) = S_2$.*

b) *Any maximal solvable subgroup of G has the form $R^* \wedge G$, where R is a maximal connected solvable subgroup of $\mathfrak{A}(G)$; and conversely.*

a) $\mathfrak{A}(S_1)$ and $\mathfrak{A}(S_2)$ are solvable connected subgroups of $\mathfrak{A}(G)$ by Proposition 25 e), defined over K, hence contained in maximal solvable connected subgroups R_1 and R_2 defined over K. By the conjugation theorem [2, Th. 16.5, p. 66, and Prop. 16.10, p. 68], there is an $s \in \mathfrak{A}(G)$, rational over K, such that $sR_1s^{-1} = R_2$; replacing S_1 by $\mathfrak{a}_s(S_1)$ (and using cor. 1 to Prop. 28), we can suppose that $R_1 = R_2$, and therefore S_1 and S_2 are contained in the solvable subgroup R^*_1 (cor. to Prop. 19). As both are maximal in G, if they were distinct, neither would be contained in the other, hence $S_1 \vee S_2$ would be distinct from S_1 and S_2; but as $S_1 \vee S_2$ is contained in the solvable group R^*_1, it would be solvable, contrary to definition.

b) If S is a maximal solvable subgroup of G, it follows as in a) that it is contained in R^*, where R is a maximal connected solvable subgroup of $\mathfrak{A}(G)$; as $R^* \wedge G$ is solvable, it must be S. Conversely, let R be a maximal connected solvable subgroup of $\mathfrak{A}(G)$; if $R^* \wedge G$ were not maximal, it would be properly contained in a maximal solvable subgroup S_1 of G, which in turn would be of the form $R^*_1 \wedge G$ where R_1 is a maximal connected solvable subgroup of $\mathfrak{A}(G)$. Then there is $s \in \mathfrak{A}(G)$, rational over K, such that $R = sR_1s^{-1}$, hence $\mathfrak{a}_s(S_1) \subset R^*$; as $\dim S_1 > \dim S$, $S \vee \mathfrak{a}_s(S_1)$, which is contained in R^*, would be a solvable subgroup of G properly containing S, contrary to the assumption that $S = R^* \wedge G$.

I have not been able to determine if the algebraic hull $\mathfrak{A}(S)$ of S in G is a maximal solvable connected subgroup of $\mathfrak{A}(G)$.[15]

If G is an arbitrary formal Lie group, S a maximal solvable subgroup of G, then S contains the center $Z(G)$; for if $\mathfrak{f}$ is the natural epimorphism of G onto $G' = G/Z(G)$, $S' = \mathfrak{f}(S)$ is solvable, and $S'' = \mathfrak{f}^{-1}(S')$ is such that $S''/Z(G)$, isogenous to S', is solvable; hence S'' is solvable, and as $S'' \supset S$, it must be equal to S. The same argument proves that S' is a maximal solvable subgroup of $G/Z(G)$ and that conversely the inverse image of a maximal solvable subgroup of $G/Z(G)$ is a maximal solvable subgroup of G. I do not know if the l.u.b. of all maximal solvable subgroups of G is equal to G; however, the derived group $D(G')$ is isogenous to a group H^*, where H is a connected algebraic matrix group (Theorem 4), and the l.u.b.

[15] This is related to another unsolved question: is the center $Z(S)$ equal to the center $Z(G)$ for a maximal solvable subgroup S (cf. [2, p. 72, prop. 18.5])?

of the maximal solvable connected subgroups of H is H itself [2, Th. 17.4, p. 71]; hence the maximal solvable subgroups of G' have a l.u.b. which contains $D(G')$ (cor. 2 to Th. 5), from which it follows that *the* l.u.b. *of the maximal solvable subgroups of* G *contains* $D(G) \vee Z(G)$. A question which remains unsettled for general formal Lie groups is whether any two maximal solvable subgroups are isomorphic; all which follows from Proposition 31 and the preceding remarks is that their quotients by $Z(G)$ are isomorphic, and therefore they have same dimension.

PROPOSITION 32. *If S is a maximal solvable subgroup of a formal Lie group G, S is equal to its normalizer in G.*

Let N be the normalizer of S in G; if $N' = N/S$ is commutative, then N is solvable, which implies $N = S$ by definition. If not, as $D(N') \neq e$, there are solvable subgroups $S' \subset N'$ not reduced to e, and the inverse image S_1 of S' in N is then solvable, since S_1/S and S are solvable; this is a contradiction, hence $N = S$.

19. Maximal tori. We will say a formal Lie group G is a *torus* if it is isomorphic to a direct product I_0^m of multiplicative groups. It follows immediately from [9, p. 236, Th. 3] that any subgroup and any quotient group of a torus is a torus.

LEMMA 3.[16] *Let N be a normal subgroup of a formal Lie group G, containing the center $Z(G) = Z$ and such that N/Z is a torus. Then $N = Z$.*

Using Proposition 4, we can suppose that the group law of G is pseudo-canonical, and that both subgroups Z and N are typical; more precisely, if m, q and n are the dimensions of Z, N and G, and if we denote by I, J and H the intervals $1 \leqq i \leqq n$, $1 \leqq i \leqq m$, $1 \leqq i \leqq q$, the X_α or Z_α of the structural basis of $\mathfrak{G}$ such that $\alpha \in \boldsymbol{N}^J$ (resp. $\alpha \in \boldsymbol{N}^H$) form the structural basis of the subhyperalgebra $\mathfrak{Z}$ (resp. $\mathfrak{N}$) of the group Z (resp. N). We first prove that if $i \in H - J$, then X_{0i} commutes with any element Y of $\mathfrak{g}_0$. As N is a normal subgroup, we have $[X_{0i}, Y] \in \mathfrak{g}_0 \cap \mathfrak{N}$ (cor. 1 to Th. 2); on the other hand, the class $\bar{X}_{0i}$ of X_{0i} modulo the normal ideal $\mathfrak{a}(\mathfrak{Z})$ is in the Lie algebra of the torus N/Z, hence we may assume it is such that $\bar{X}^p_{0i} = \bar{X}_{0i}$ [9, p. 233, prop. 5]; this means that $X^p_{0i} - X_{0i}$ belongs to the intersection of $\mathfrak{a}(\mathfrak{Z})$ and $\mathfrak{g}_0$, i.e. to $\mathfrak{g}_0 \cap \mathfrak{Z}$, which is contained in the center of the Lie algebra $\mathfrak{g}_0$. Hence we have $[X^p_{0i} - X_{0i}, Y] = 0$, or

$$[X_{0i}, Y] = [X^p_{0i}, Y] = [X_{0i}, [X_{0i}, [\cdots [X_{0i}, Y] \cdots]]]$$

[16] **Lemma 3 and its corollary are not true for formal Lie groups over a field of characteristic 0.**

by Jacobson's formula. Now, N/Z being commutative, we see as above that $[U, V] \in \mathfrak{g}_0 \cap \mathfrak{Z}$ for any two elements of the Lie algebra $\mathfrak{g}_0 \cap \mathfrak{N}$; the previous formula then proves, first that $[X_{0i}, Y]$ is in $\mathfrak{g}_0 \cap \mathfrak{Z}$, and then that it is 0.

As a basis for induction, suppose now we have proved that all elements X_{hi} such that $h < r$ and $i \in H - J$ are in the center of the algebra $\mathfrak{s}_{r-1}$; the same is true therefore for all X_α of height $h(\alpha) < r$, such that $\alpha \in \boldsymbol{N}^H$, and also (by Lemma 1) for the Z_α of corresponding indices. We now proceed in several steps.

(i) We first prove that $[X_{ri}, Y] = [X^p_{ri}, Y]$ *for* $i \in H - J$, $Y \in \mathfrak{s}_{r-1}$. The difference $X^p_{ri} - X_{ri}$ belongs to the Lie-p-algebra $\mathfrak{g}_r \cap \mathfrak{N}$, and applying to it the iterated homomorphism $\boldsymbol{p}'^r$ shows that $X^p_{ri} - X_{ri} = S + T$, where $S \in \mathfrak{g}_r \cap \mathfrak{Z}$ and $T \in \mathfrak{s}_{r-1} \cap \mathfrak{N}$; the result follows from the fact that $[S, Y] = 0$ by definition and $[T, Y] = 0$ by the inductive hypothesis.

(ii) Next we show that $[X_{ri}, X_{0j}] = 0$ *for* $i \in H - J$ *and for any* $j \in I$. The inductive hypothesis first proves, as in the proof of Theorem 3, that $[X_{ri}, X_{0j}]^0 - 1 \otimes [X_{ri}, X_{0j}] - [X_{ri}, X_{0j}] \otimes 1 = 0$, i.e. that $[X_{ri}, X_{0j}]$ *is in* $\mathfrak{g}_0$; on the other hand, $[X_{ri}, X_{0j}]$ is in the normal ideal $\mathfrak{a}(\mathfrak{N})$, hence in $\mathfrak{g}_0 \cap \mathfrak{N}$. But then $[X^p_{ri}, X_{0j}]$ is in the normal ideal $\mathfrak{a}(\mathfrak{Z})$ by Jacobson's formula and the commutativity of N/Z, and on the other hand it is in $\mathfrak{g}_0$ by (i) and by what has just been proved; therefore $[X_{ri}, X_{0j}] \in \mathfrak{g}_0 \cap \mathfrak{Z}$, and then Jacobson's formula yields $[X^p_{ri}, X_{0j}] = 0$, which proves our contention by (i).

(iii) We can now prove that $[X_{ri}, Y] = 0$ *for* $i \in H - J$, $Y \in \mathfrak{s}_{r-1}$. We can of course suppose $Y = Z_\alpha$ with $h(\alpha) < r$, and we use induction on $|\alpha|$, the result being proved in (ii) for $|\alpha| = 1$. As in (ii), the induction hypothesis first shows that $[X_{ri}, Z_\alpha]^0 - 1 \otimes [X_{ri}, Z_\alpha] - [X_{ri}, Z_\alpha] \otimes 1 = 0$, hence $[X_{ri}, Z_\alpha] \in \mathfrak{g}_0$. Then $[X^p_{ri}, Z_\alpha] = 0$ by (ii) and Jacobson's formula, and we conclude by (i).

(iv) The next step is to show that $[X_{0i}, X_{rj}] = 0$ *for* $i \in H - J$ *and any* $j \in I$. As we have $X^p_{0i} - X_{0i} \in \mathfrak{Z}$, $[X_{0i}, X_{rj}] = [X^p_{0i}, X_{rj}]$; on the other hand, the inductive hypothesis proves as above that $[X_{0i}, X_{rj}] \in \mathfrak{g}_0$; Jacobson's formula then yields the result.

(v) Now we prove by induction on $h < r$, that $[X_{hi}, X_{rj}] = 0$ *for* $i \in H - J$ *and any* $j \in I$. From Lemma 1 and the inductive hypothesis, it follows that $[Z_\alpha, X_{rj}] = 0$ for $\alpha \in \boldsymbol{N}^H$ and $h(\alpha) < h$. As before, this implies that $[X_{hi}, X_{rj}] \in \mathfrak{g}_0$. But as in (i) we remark that $X^p_{hi} - X_{hi} = S + T$, where $S \in \mathfrak{g}_h \cap \mathfrak{Z}$ and $T \in \mathfrak{s}_{h-1} \cap \mathfrak{N}$; the inductive hypothesis then proves that $[X_{hi}, X_{rj}] = [X^p_{hi}, X_{rj}]$, and the conclusion follows as usual from Jacobson's formula.

(vi) We can now take the final step and prove that $[X_{ri}, X_{rj}] = 0$ *for* $i \in H - J$ *and any* $j \in I$. From (ii) and (v) it follows that X_{ri} commutes with all elements of $\mathfrak{s}_{r-1}$ and X_{rj} with all elements of $\mathfrak{s}_{r-1} \cap \mathfrak{N}$; the usual argument implies that $[X_{ri}, X_{rj}] \in \mathfrak{g}_0$. On the other hand, as $X^p_{ri} - X_{ri} = S + T$, with $S \in \mathfrak{g}_r \cap \mathfrak{Z}$ and $T \in \mathfrak{s}_{r-1} \cap \mathfrak{N}$, we have $[X_{ri}, X_{rj}] = [X^p_{ri}, X_{rj}]$ by (v), and (ii) and Jacobson's formula end the proof. The induction on r is thus completed, and shows that every element of the hyperalgebra $\mathfrak{N}$ commutes with every element of $\mathfrak{G}$, which concludes the proof of Lemma 3.

COROLLARY. *A torus N which is a normal subgroup of G is contained in the center.*

For $(N \vee Z)/Z$ is isogenous to a quotient group of N (Proposition 11), hence a torus, and $N \vee Z$ is a normal subgroup of G, so that we can apply Lemma 3.

PROPOSITION 33. *If G is a torus in $GL^*(n)$ over an algebraically closed field, then $\mathcal{A}(G)$ is a torus in $GL(n)$.*

From Proposition 25 e), we know that $\mathcal{A}(G)$ is commutative and connected, and by [2, p. 53, Th. 11.1] $\mathcal{A}(G)$ is the direct product of a torus C_s and a unipotent abelian group C_u. Let v be the (rational) projection of $\mathcal{A}(G)$ onto C_u, with kernel C_s; then the corresponding homomorphism v of $\mathcal{A}^*(G)$ onto C^*_u has kernel C^*_s (Proposition 18), and as C^*_u is unipotent, $v(G) = e$ [9, p. 240, Th. 4], which proves that $G \subset C^*_s$, in other words $\mathcal{A}(G) = C_s$, q. e. d.

PROPOSITION 34. *Let G be a subgroup of $GL^*(n)$, over an algebraically closed field K. Then:*

a) *If T_1, T_2 are any two maximal tori of G, there is an $s \in \mathcal{A}(G)$ rational over K, such that $a_s(T_1) = T_2$.*

b) *Any maximal torus of G has the form $R^* \wedge G$, where R is a maximal torus in $\mathcal{A}(G)$; and conversely.*

c) *Any maximal torus in a maximal solvable subgroup of G is a maximal torus in G.*

The proofs of a) and b) run exactly parallel to the proofs in Proposition 31, using Proposition 33 and [2, cor. 16.6, p. 67] instead of [2, Th. 16.5]; they can therefore be omitted. Statement c) follows at once from b) and from the corresponding statement in [2, cor. 16.6].

Here again, it remains to be seen if, for a maximal torus T of G, $\mathcal{A}(T)$ is a maximal torus in $\mathcal{A}(G)$ (cf. cor. to Proposition 38). For a general

formal Lie group G, let T be a maximal torus of $G/Z(G)$; if we apply Lemma 3 to the group $f^{-1}(T)$, where f is the natural epimorphism of G onto $G/Z(G)$, we see that $f^{-1}(T)$ is *commutative*, hence is the product of a unique torus T_0, its *core* [9, p. 236, Th. 3], and of its p-radical U; moreover, U is contained in $Z(G)$ by [9, p. 240, Th. 4], and $T=f(T_0)$. Conversely, if T'_0 is any maximal torus in G, $f(T'_0)$ is a torus in $G/Z(G)$, and if it was properly contained in a torus T in $G/Z(G)$, the preceding argument would show that T'_0 is properly contained in a torus in $f^{-1}(T)$. We therefore deduce from Proposition 34 that *all maximal tori in a formal Lie group G over an algebraically closed field are isomorphic.* Furthermore:

PROPOSITION 35. *In a formal Lie group G over an algebraically closed field, the intersection of all maximal tori of G is the core of the center of G.*

From the preceding remarks, and Proposition 34 b), we see that the intersection C of the maximal tori of G is contained in $f^{-1}(D^* \wedge G')$, where f is a homomorphism of G into a group $GL^*(n)$, with kernel $Z(G)$, and D the intersection of all maximal tori of the group $\mathcal{C}(G')$. But, by [2, p. 71, Prop. 18.1], D is a torus contained in the center of $\mathcal{C}(G')$, hence $D^* \wedge G'$ is a torus contained in the center of G', and by Lemma 3, $f^{-1}(D^* \wedge G')$ is contained in the center of G, q. e. d.

20. Nilpotent groups and Cartan subgroups. We say that a formal Lie group G is *unipotent* if there exists a power p^k of the characteristic such that $x^{p^k}=e$ (which means $\phi^{(p^k)}(x)=e$, if we define, by induction on h, the system $\phi^{(h)}(x)$ of n power series as equal to $\phi(x, \phi^{(h-1)}(x))$).

PROPOSITION 36. *Any unipotent formal Lie group G is nilpotent.*

It is clear that the representable group $G'=G/Z(G)$ is unipotent. If u is a monomorphism of G' into a group $GL^*(m)$, and z a generic point for G', the matrix $u(z)$ is unipotent. As the set of unipotent matrices in $GL(m)$ is defined by a system of algebraic equations, we conclude that $\mathcal{C}(G')$ is a group of unipotent matrices, hence is nilpotent [2, p. 74, Th. 19.4]. Therefore G' is nilpotent, and so is of course G.

PROPOSITION 37. *A representable nilpotent group G over an algebraically closed field is quasi-direct product of its unique maximal torus and of its unique maximal unipotent subgroup.*

We can suppose G is a subgroup of a group $GL^*(n)$. Then $\mathcal{C}(G)$ is nilpotent, hence [2, p. 53, Th. 11.1] the direct product of its unique maximal

torus C and its unique maximal unipotent subgroup U. It is clear that $D(\mathfrak{A}(G)) = D(U)$. The group $D^*(U)$ is equal to $D(G)$ (Theorem 4), and the abelian group $G/D(G)$ is thus contained in the group $C^* \times (U^*/D(U^*))$. Now $G/D(G)$ is the direct product of its core H and its p-radical R, and as C^* is a torus and $U^*/D(U^*)$ a unipotent abelian group, we have $H \subset C^*$ and $R \subset U^*/D(U^*)$ [9, p. 240, Th. 4], which shows in particular that R is a unipotent group. The inverse image V of R in G contains $D(G) = D(U^*)$, and as R and $D(U^*)$ are unipotent, so is V, and therefore, by the same argument as in Proposition 36, we have $V \subset U^*$; moreover $V = G \wedge U^*$, since $V \subset G \wedge U^*$, and if this was a proper inclusion, the dimension of the p-radical of $G/D(G)$ would be strictly greater than that of R. A similar argument shows that if f is the projection of $\mathfrak{A}(G)$ onto U, $f(G) = V$, hence G is contained in the direct product $V \times C^*$, and as it contains V, it is equal to $V \times (G \wedge C^*)$; as $G \wedge C^* = T$ is the maximal torus of G (Proposition 34 b)), the proof is complete.

Proposition 37 applies in particular to *abelian* representable groups; it shows at once that (in contrast with Ado's theorem for formal Lie groups over a field of characteristic 0) there exist *non representable* abelian groups. This is for instance the case for all one-dimensional groups I_r $(0 < r < +\infty)$ [9, p. 229, Th. 2], for I_0 is the only one-dimensional torus and $W_1 = I_\infty$ (the additive group) the only unipotent one-dimensional group. Similarly, all *simple* abelian groups of dimension > 1 [11, pp. 450-451] are non-representable, by Propositions 36 and 37; this shows incidentally that if a simple abelian group S of dimension > 1 is contained in a formal Lie group G, it is contained *in the center* of G. It is clear that if two subgroups H_1, H_2 of a formal Lie group G are such that there is *no* homomorphism of H_1 nor of H_2 into a group $GL^*(n)$ which is not trivial (such groups may be called *totally nonrepresentable groups*, and the previous examples are of that kind), then $H_1 \vee H_2$ has the same property. There is therefore a *largest* totally non-representable group N, contained of course in the center of G. It would be interesting to know the structure of N and if N is a quasi-direct factor of G.

THEOREM 6. *In order that a formal Lie group G over an algebraically closed field K be nilpotent, it is necessary and sufficient that it contain a unique maximal torus, which is then in the center of G.*

a) The *necessity* of the condition follows from Proposition 37 if G is representable. In general, $G/Z(G)$ is nilpotent with G, hence contains a unique maximal torus T which is in its center. But as T is normal in $G/Z(G)$, its inverse image $f^{-1}(T)$ by the natural mapping f of G onto

$G/Z(G)$ is a normal subgroup of G containing $Z(G)$ and whose quotient by $Z(G)$ is a torus; hence $f^{-1}(T)=Z(G)$ by Lemma 3, which means T is reduced to e, and the unique maximal torus of G is the core of $Z(G)$.

b) To prove *sufficiency*, let us first assume G is a subgroup of a group $GL^*(n)$, and let T be the unique maximal torus of G. Then, by Proposition 34a), for any $s\in\mathfrak{A}(G)$, rational over K, $\mathfrak{a}_s(T)=T$; as the points of $\mathfrak{A}(G)$ which are rational over K are dense in $\mathfrak{A}(G)$, we see that $\mathfrak{A}(G)$ is contained in $\mathfrak{N}(T)$, and therefore T is a normal subgroup in $\mathfrak{A}^*(G)$ (and of course in G). From the corollary to Lemma 3, we conclude that T is in the center of $\mathfrak{A}^*(G)$, hence $\mathfrak{A}(T)$ is in the center of $\mathfrak{A}(G)$ by Proposition 25b). Furthermore, from Proposition 34b), any maximal torus S of $\mathfrak{A}(G)$ is such that $S^*\wedge G=T$, and this implies that $S\supset\mathfrak{A}(T)$, otherwise the connected component of $S\cap\mathfrak{A}(T)$ would have strictly smaller dimension than $\mathfrak{A}(T)$ and contain T, contrary to the definition of $\mathfrak{A}(T)$. There is a rational homomorphism g of $\mathfrak{A}(G)$ into a group $GL(m)$, with kernel $\mathfrak{A}(T)$ [2, p. 40, Lemma 5.10.2], and the maximal tori of the algebraic group $H=g(\mathfrak{A}(G))$ are the images $g(S)$ where S is a maximal torus of $\mathfrak{A}(G)$ [2, p. 80, Th. 22.1]. If $G'=\mathfrak{g}(G)$, H is equal to $\mathfrak{A}(G')$, otherwise the inverse image of $\mathfrak{A}(G')$ by g would contain G and be properly contained in $\mathfrak{A}(G)$, contrary to definition. Furthermore, we have $G'\wedge\mathfrak{g}(S^*)=e$; indeed $G'\vee\mathfrak{g}(S^*)=\mathfrak{g}(G\vee S^*)$ and as $G\vee S^*$ contains the kernel $\mathfrak{A}^*(T)$ of $\mathfrak{g}$, we have $\dim(G'\vee\mathfrak{g}(S^*))=\dim(G\vee S^*)-\dim\mathfrak{A}^*(T)$. But as $G\wedge S^*=T$, $\dim(G\vee S^*)=\dim G+\dim S^*-\dim T$ by Proposition 11 (G being normal in $\mathfrak{A}^*(G)$), hence

$$\begin{aligned}\dim(G'\vee\mathfrak{g}(S^*)) &= (\dim G-\dim T)+(\dim S^*-\dim\mathfrak{A}^*(T))\\ &= \dim G'+\dim\mathfrak{g}(S^*),\end{aligned}$$

which proves by Proposition 11 that $\dim(G'\wedge\mathfrak{g}(S^*))=0$ (G' being normal in H). Now it is enough to prove that G' is nilpotent, since T is in the center of G; the preceding argument thus allows us to restrict ourselves to the case in which $T=e$.

We are actually going to prove that in such a case $\mathfrak{A}(G)$ (hence also G) is a *unipotent* group. First there is no maximal torus in $D(G)$, which by Theorem 4 is equal to $(D(\mathfrak{A}(G))^*$; by [2, p. 67, cor. 16.7], the algebraic group $D(\mathfrak{A}(G))$ is unipotent. Let f be a rational homomorphism of $\mathfrak{A}(G)$ into a group $GL(m)$, with kernel $D(\mathfrak{A}(G))$ [2, p. 40, Lemma 5.10.2]; it will be enough to prove that the algebraic abelian group $H=f(\mathfrak{A}(G))$ is unipotent. Suppose the contrary: then, by [2, p. 53, Th. 11.1], H is the

direct product of its unique maximal torus C and its unique maximal unipotent subgroup U, and by assumption $C \neq \{e\}$. Furthermore, the abelian formal Lie group $\boldsymbol{f}(G)$ is a subgroup of H^*, and a direct product of its core $Q \subset C^*$ and its p-radical $R \subset U^*$ [9, p. 240, Th. 4]. If we had $Q = \boldsymbol{e}$, $\boldsymbol{f}(G)$ would be contained in U^*, hence G would be contained in the algebraic group $f^{-1}(U)$, which is properly contained in $\mathcal{A}(G)$, contrary to the definition of $\mathcal{A}(G)$. We have therefore $\dim Q > 0$. Consider now the algebraic subgroup $C_0 = f^{-1}(C)$ of $\mathcal{A}(G)$, which is such that C_0^* contains the normal subgroup $Q_0 = \boldsymbol{f}^{-1}(Q)$ of G. If S is a maximal torus of H, $f(S)$ is a maximal torus of H [2, p. 80, Th. 22.1] hence equal of C, which proves that $S \subset C_0$. We are going to show that $S^* \wedge Q_0$ is not reduced to $\boldsymbol{e}$, which is a contradiction by the assumption and by Proposition 34b). As we have $\boldsymbol{f}(S^* \vee Q_0) = C^* \vee Q = C^*$, we have $\dim C^* = \dim(S^* \vee Q_0) - \dim D(G) = \dim S^*$, since $S^* \wedge D(G) = \boldsymbol{e}$. This, by Proposition 11, yields $\dim(S^* \wedge Q_0) = \dim Q_0 - \dim D(G) = \dim Q > 0$, and we have reached the desired conclusion.

Suppose now G is an arbitrary formal Lie group having a unique maximal torus T; then, if $\boldsymbol{u}$ is the natural epimorphism of G onto $G/Z(G)$, $\boldsymbol{u}(T)$ is the unique maximal torus T_0 of $G/Z(G)$, and by the preceding argument, $G/Z(G)$ is nilpotent, hence also G. Furthermore, $\boldsymbol{u}^{-1}(T_0)$ is normal in G and its quotient by $Z(G)$ is a torus; therefore Lemma 3 proves that actually $T_0 = \boldsymbol{e}$, $T \subset Z(G)$ and $G/Z(G)$ is *unipotent*. I do not know if T is then a quasi-direct factor of G.

PROPOSITION 38. *Let G be a representable solvable formal Lie group over an algebraically closed field. Then there is a largest unipotent subgroup G_u of G, which is normal in G, and for any maximal torus T of G, $G = T \vee G_u$ and $G_u \wedge T = \boldsymbol{e}$.*

We suppose $G \subset GL^*(n)$. Consider the derived group $D(G)$, which is nilpotent (Proposition 30) and equal to $(D(\mathcal{A}(G)))^*$ (Theorem 4). There is a rational homomorphism f of $\mathcal{A}(G)$ into a group $GL(m)$, with kernel $D(\mathcal{A}(G))$, and $\boldsymbol{f}(G)$ is an abelian representable group, which is therefore direct product of its core C and its p-radical R, which is unipotent (Proposition 37). Let $N = \boldsymbol{f}^{-1}(R) \supset D(G)$; this is a solvable normal subgroup of G. Moreover, if S is a maximal torus in N, we have $\boldsymbol{f}(S) = \boldsymbol{e}$, and $S \subset D(G)$. But as $D(G)$ is nilpotent, it has a unique maximal torus, and so therefore has N, which proves, by Theorem 6, that N is nilpotent. N is therefore, by Proposition 37, quasi-direct product of its largest unipotent subgroup G_u and its unique maximal torus S. Furthermore, if V is a

unipotent subgroup of G, and $\boldsymbol{g}$ the natural homomorphism of G onto G/N (which is isogenous to C, hence a torus), we have $\boldsymbol{g}(V) \subset G/N$, hence as $\boldsymbol{g}(V)$ is unipotent, $\boldsymbol{g}(V) = \boldsymbol{e}$, $V \subset N$, and finally $V \subset G_u$; and as G_u is the largest unipotent subgroup of G, it is normal in G. Let now T be any maximal torus of G; $T \wedge N$ is a torus in N, hence contained in S; on the other hand, S being the largest torus in the normal subgroup N, is a normal subgroup of G. In the group $T \vee S$, S is a normal subgroup such that $(T \vee S)/S$ is a torus (isogenous to $T/(T \wedge S)$), and Lemma 3 and its corollary prove that $T \vee S$ is a torus, hence $S \subset T$ by definition. Furthermore we have $T \wedge N = S$ and $T \wedge G_u = \boldsymbol{e}$; finally $\boldsymbol{f}(T)$ is contained in C, hence equal to C for reasons of dimension, hence $\boldsymbol{g}(T) = G/N$, $T \vee G_u = G$.

COROLLARY. *The algebraic group $\mathfrak{A}(T)$ is a maximal torus in $\mathfrak{A}(G)$, and $\mathfrak{A}(G_u)$ is the largest unipotent subgroup of $\mathfrak{A}(G)$.*

Indeed, $\mathfrak{A}(G_u)$ is a normal unipotent subgroup of $\mathfrak{A}(G)$, $\mathfrak{A}(T)$ a torus in $\mathfrak{A}(G)$, hence $\mathfrak{A}(T) \cdot \mathfrak{A}(G_u)$ is an algebraic subgroup of $\mathfrak{A}(G)$ which obviously contains G, hence $\mathfrak{A}(G) = \mathfrak{A}(T) \cdot \mathfrak{A}(G_u)$; as the intersection of a torus and a unipotent subgroup only contains the identity, this proves our assertion.

Note that if R is a maximal solvable subgroup of a representable formal Lie group G, then the largest unipotent subgroup of R is also a maximal unipotent subgroup of G, as follows from the conjugation theorem (Proposition 31).

PROPOSITION 39. *In a formal Lie group G over an algebraically closed field, let T be a maximal torus. The normalizer C of T is also the centralizer of T, and it is a maximal nilpotent subgroup of G, equal to its normalizer in G; moreover, any maximal solvable subgroup of G containing T contains C. Conversely, any nilpotent subgroup H of G which is equal to its normalizer in G is the centralizer of a maximal torus of G.*

We closely follow the proofs of the corresponding results of A. Borel [2, p. 67, cor. 16.8 and p. 76, Lemma 20.6]. It follows at once from the corollary to Lemma 3 that C is also the centralizer of T in G; moreover, if N is the normalizer of C in G, T being the unique maximal torus of the normal subgroup C of N, is a normal subgroup of N, hence $N = C$. From the uniqueness of the maximal torus in C it follows (Theorem 6) that C is nilpotent. C is contained in a maximal solvable subgroup R of G; if G is representable, it follows from that remark and from the conjugation theorems (Propositions 31 and 34) that any maximal solvable subgroup of G con-

taining T contains C. That result is easily extended to the general case, remembering the one-to-one correspondence between maximal solvable subgroups (resp. maximal tori) of G and of $G/Z(G)$. Finally, if Q is a nilpotent subgroup containing C, T is the unique maximal torus of Q, hence in the center of Q, which proves that $Q = C$.

To prove the converse, let us first suppose G is a subgroup of $GL^*(n)$. Then H is direct product of its unique maximal torus S and its largest unipotent subgroup H_u (Proposition 37). Consider a maximal solvable subgroup R of G containing H, and let Q be a maximal torus of R containing S. Let M be the centralizer of S *in* R, which contains H and Q; the largest unipotent subgroup M_u of the solvable group M then contains H_u and we have $M = M_u \vee Q$ by Proposition 38. If we had $H_u \neq M_u$, the normalizer N of H_u in M_u would be different from H_u; this is seen by the same argument as in [2, p. 75, Lemma 20.3], which we suppress. But then, if $\boldsymbol{f}$ is the natural mapping of M onto M/S, $\boldsymbol{f}(H_u)$ would be normal in $\boldsymbol{f}(N)$ and distinct from $\boldsymbol{f}(N)$ for reasons of dimension, since $S \wedge N = \boldsymbol{e}$; hence, by Proposition 11, $H = S \vee H_u$ would be normal in $S \vee V$ and distinct from that group, contrary to assumption. Therefore $H_u = M_u$, and as M/M_u is abelian (Proposition 38), H contains the derived group $D(M)$, hence is normal in M; but the hypothesis then implies $H = M$, hence $Q = S$, and S is equal to a maximal torus of R, hence of G (Proposition 34c)). If now C is the centralizer of S *in* G, it is the direct product of S by its largest unipotent subgroup $C_u \supset H_u$. If we had $H_u \neq C_u$, it would follow as above that H would be distinct from its normalizer in G, hence $H = C$, and the proof is complete in that case.

Finally, if G is not representable, it is clear that a subgroup of G equal to its normalizer must contain the center $Z(G)$; but in the one-to-one correspondence between subgroups of $G/Z(G)$ and subgroups of G containing $Z(G)$, nilpotent groups correspond to nilpotent groups and normalizers to normalizers: the image of H in $G/Z(G)$ is thus the normalizer (or centralizer) of a maximal torus T of $G/Z(G)$, and as we know that T is the image of a maximal torus T_0 of G, H is the normalizer of T_0.

We say that the centralizers of maximal tori in G are the *Cartan subgroups* of G.

COROLLARY. *Let G be a subgroup of $GL^*(n)$, over an algebraically closed field K. Then, if C_1, C_2 are any two Cartan subgroups of G, there is an $s \in \mathcal{C}(G)$, rational over K, such that $\boldsymbol{a}_s(C_1) = C_2$.*

This follows immediately from Proposition 34a).

If G is a subgroup of $GL^*(n)$, over an algebraically closed field, the

relation between the algebraic hull $\mathfrak{A}(C)$ of a Cartan subgroup C of G and the Cartan subgroups of $\mathfrak{A}(G)$ is not clear; nor do we know if the l. u. b. of all Cartan subgroups of G is equal to G.

21. Semisimple and simple formal Lie groups. We say that a formal Lie group G is *semisimple* if it contains no abelian normal subgroup distinct from the identity. A *simple* Lie group is obviously semisimple if it is not abelian.

THEOREM 7. *If G is a semisimple* (resp. *non abelian simple*) *Lie group over a field K, then G is isogenous to a group G'^*, where G' is a semisimple* (resp. *non abelian simple*) *algebraic group of matrices defined over K. Conversely, if G' is a semisimple* (resp. *non abelian simple*) *algebraic group of matrices, G'^* is a semisimple* (resp. *non abelian simple*) *formal Lie group.*

If G is a semisimple Lie group, $Z(G)=\boldsymbol{e}$ by definition, hence G is representable. Among all isogenies $\boldsymbol{u}$ of G into groups $GL^*(n)$, consider one for which the dimension of $\mathfrak{A}(\boldsymbol{u}(G))$ is *minimum*. If G is semisimple, $\mathfrak{A}(\boldsymbol{u}(G))$ cannot have an algebraic abelian normal subgroup N of dimension >0 defined over K; for there would then be a rational homomorphism f of $\mathfrak{A}(\boldsymbol{u}(G))$ into a group $GL(m)$, defined over K, with kernel N [2, p. 40, Lemma 5.10.2]; as $N^* \wedge \boldsymbol{u}(G)$ is an abelian normal subgroup of $\boldsymbol{u}(G)$, it must be reduced to $\boldsymbol{e}$, hence the restriction of $\boldsymbol{f}$ to $\boldsymbol{u}(G)$ is an isogeny, and $\boldsymbol{f}\circ\boldsymbol{u}$ would be an isogeny of G into $GL^*(m)$, such that the dimension of $\mathfrak{A}(\boldsymbol{f}(\boldsymbol{u}(G)))$ would be at most $\dim \mathfrak{A}(\boldsymbol{u}(G)) - \dim N < \dim \mathfrak{A}(\boldsymbol{u}(G))$, which contradicts our choice of $\boldsymbol{u}$. The same argument proves that if G is simple and non abelian, $\mathfrak{A}(\boldsymbol{u}(G))$ has no algebraic normal subgroup of dimension >0, defined over K, other than itself. Moreover, in this last case, $D(\mathfrak{A}(G)) = \mathfrak{A}(G)$, hence, by Theorem 4, $\boldsymbol{u}(G) = \mathfrak{A}^*(G)$. The same argument holds also in the case of a semisimple group G, as it follows from results of Chevalley [7] that an algebraic connected semisimple group is equal to its derived subgroup.

Suppose conversely G' is a semisimple connected algebraic group of matrices. If G'^* contained an abelian normal subgroup N of dimension >0, $\mathfrak{A}(N)$ would be an abelian normal subgroup of G' of dimension >0 (Proposition 23), hence $N=\boldsymbol{e}$, G'^* is semisimple. Similarly, if G' is simple, and G'^* contains a normal subgroup N of dimension >0, $\mathfrak{A}(N)$ is a normal subgroup of G' of dimension >0, hence equal to G'. But, by Theorem 4, $D(N)$ is then equal to $(D(G'))^*$, hence to G'^*, and this proves that $N=G'^*$, in other words G'^* is a simple formal Lie group.

C. Chevalley has recently succeeded in determining all algebraic semi-simple groups of matrices over an algebraically closed field [7]; in particular, it follows from his results that the simple algebraic groups over such a field are those of the Cartan-Killing classification. This fundamental theorem, joined to Theorem 7, shows that all simple nonabelian formal Lie groups over an algebraically closed field are known *up to an isogeny*. It remains to determine what are the possible isogenies between simple groups; when the Lie algebra is simple, only the trivial isogenies $\boldsymbol{p}^t$ exist [8, p. 115, Theorem 7]; but it is known that some of the Lie algebras of the simple algebraic groups are not simple for certain values of the characteristic [14], and it seems likely that this phenomenon corresponds to the existence of non-trivial isogenies. I am not at present in a position to prove this conjecture in every case (actually, the structure of the Lie algebras of the groups of types E_6, E_7, E_8 and F_4 is not even yet determined over a field of arbitrary characteristic), and all I will do here is to show that it is substantiated [17] in the case of the groups of type A_n.

Let us consider the Lie algebra $\mathfrak{gl}(n)$ of the group $GL^*(n)$, and write X_{ij} its basis elements, instead of $X_{0,ij}$ $(1 \leqq i, j \leqq n)$. We have on one hand in $GL^*(n)$ the unimodular group $SL^*(n)$, which is the kernel of the homomorphism f, defined by $f((x_{ij})) = \det(\delta_{ij} + x_{ij}) - 1$ of $GL^*(n)$ into the multiplicative group I_0; it is readily verified that the Lie algebra $\mathfrak{sl}(n)$ of $SL^*(n)$ is the subalgebra of $\mathfrak{gl}(n)$ consisting of the elements $\sum_{i,j} \xi_{ij} X_{ij}$ such that $\sum_{i=1}^{n} \xi_{ii} = 0$. On the other hand, the center Z of $GL^*(n)$ is the image of I_0 by the homomorphism $\boldsymbol{g}$ such that $g_{ij}(x) = \delta_{ij} x$ $(1 \leqq i, j \leqq n)$ (Proposition 21); its Lie algebra $\mathfrak{z}$ is the subalgebra of $\mathfrak{gl}(n)$ generated by the element $X_{11} + \cdots + X_{nn}$. Now it is clear that Z is not contained in $SL^*(n)$, hence $Z \wedge SL^*(n) = \boldsymbol{e}$ since Z is one-dimensional, and from Proposition 11 it follows that $Z \vee SL^*(n) = GL^*(n)$; therefore $GL^*(n)$ is the quasi-direct product of its normal subgroups Z and $SL^*(n)$ (Proposition 13). If n is not a multiple of the characteristic p, $\mathfrak{gl}(n)$ is the direct sum of $\mathfrak{z}$ and $\mathfrak{sl}(n)$, hence (Propositions 1 and 13) $GL^*(n)$ is the direct product of Z and $SL^*(n)$. But if n is a multiple of p, we have $\mathfrak{z} \subset \mathfrak{sl}(n)$, and the restriction to $SL^*(n)$ of the natural mapping of $GL^*(n)$ onto the quotient group $PGL^*(n) = GL^*(n)/Z$ is a *non trivial isogeny* $\boldsymbol{u}$. More precisely, $\mathfrak{z}$ is the kernel of $\boldsymbol{u}'$ in $\mathfrak{sl}(n)$, and it is known [14] that the quotient Lie-p-algebra $\mathfrak{sl}(n)/\mathfrak{z}$ is *simple* unless $p = n = 2$. From the homo-

[17] This example has been pointed out to me by J. P. Serre.

morphism theorem it then follows that the Lie algebra of $PGL^*(n)$ has an (n^2-2)-dimensional ideal isomorphic to $\mathfrak{sl}(n)/\mathfrak{z}$ [8, p. 116]. This shows that the Lie algebras of $SL^*(n)$ and $PGL^*(n)$ *are not isomorphic* when p divides n and $n \neq 2$; direct computation leads to the same conclusion when $n=2=p$. This provides examples to some situations discussed earlier: $\boldsymbol{u}$ is composed of the natural epimorphism of $GL^*(n)$ onto $PGL^*(n)$ and of the natural monomorphism of $SL^*(n)$ into $GL^*(n)$, but fails to be separable (cf. no. 4); the example $F=SL^*(n)$, $N=Z$ shows that $F/(F \wedge N)$ is not always isomorphic to $(F \vee N)/N$ (no. 10); and the two Jordan-Hölder sequences $GL^*(n) \supset SL^*(n) \supset \boldsymbol{e}$, $GL^*(n) \supset Z \supset \boldsymbol{e}$ have quotient groups which are *not* isomorphic (no. 10).

As a last remark, let us observe that it follows from Chevalley's result mentioned above that the simple Lie-p-algebras over an algebraically closed field, discovered and studied in recent years by several authors (see e.g. [1] and [15]), which are not isomorphic to Lie algebras of the groups of the Cartan-Killing classification, *cannot correspond to formal Lie groups.* For some of these algebras it is possible to see this directly without using Chevalley's deep results. For instance, consider the *Witt algebra* $\mathfrak{w}$ (the first one to be discovered) over an algebraically closed field K of characteristic $p>0$: it is defined as the algebra of derivations of the associative algebra $\mathfrak{A}=K[X]/(X^p)$ over K and has dimension p. It is known [15] that $\mathfrak{w}$ is simple if $p>2$, and that its group of automorphisms is *solvable* if $p \geqq 5$, But if $\mathfrak{w}$ was the Lie algebra of a formal Lie group G, G would be simple and non abelian [8, p. 115, Theorem 7], hence of the form G'^* by Theorem 7, where G' is an algebraic simple connected group of matrices. Consider now the automorphisms $\boldsymbol{a}_s$ of G, where s ranges over the rational points of G' over K; the restriction $\boldsymbol{f}_s$ of $\boldsymbol{a}'_s$ to the Lie algebra $\mathfrak{g}_0=\mathfrak{w}$ of G would be an automorphism of $\mathfrak{w}$, and $s \to \boldsymbol{f}_s$ a rational homomorphism (over K) of the group Γ of all elements of G' rational over K, into the group L of all automorphisms of $\mathfrak{w}$. Now L is an algebraic matrix K-group, and so is Γ; the homomorphism $s \to \boldsymbol{f}_s$ gives rise to a homomorphism of the Lie algebra $\mathfrak{w}$ of Γ into the Lie algebra $\mathfrak{l}$ of L, and the same argument as in [5, p. 122] shows that this is identical to the usual "adjoint mapping" $X \to \mathrm{ad}(X)$ of $\mathfrak{w}$ into $\mathfrak{l}$, with $\mathrm{ad}(X) \cdot Y = [X, Y]$. As $\mathfrak{w}$ is simple, the kernel of the mapping $X \to \mathrm{ad}(X)$ is reduced to 0, hence, by the homomorphism theorem, the kernel N of $s \to \boldsymbol{f}_s$ is a finite normal subgroup of Γ; therefore, for $p \geqq 5$, Γ/N would be a solvable group. As G is isomorphic to $(\Gamma/N)^*$, G would be solvable, which is absurd.

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ON MEAN MOTION.*

By RAOUF DOSS.

Introduction. Let $p(t) = \sum_1^N a_n \exp(i\lambda_n t)$ be a trigonometric polynomial. The Lagrange *mean motion* problem is to prove that the argument $\phi(t)$ of $p(t)$ has a mean value c, i.e., that there exists a limit

$$c = \lim[\phi(\delta) - \phi(\gamma)]/(\delta - \gamma) \text{ as } (\delta - \gamma) \to \infty.$$

The existence of the limit c has been stated in Jessen [4] and proved by Jessen and Tornehave in their paper [5]. We refer to this important paper for a full treatment of the Lagrange and allied problems and for bibliography.

Since $\arg p(t)$ is not determined at a zero of $p(t)$, we shall denote by $\arg^- p(t)$, viz. $\arg^+ p(t)$ any branch of $\arg p(t)$ which is continuous, except at a zero of $p(t)$, and discontinuous with a jump $-p\pi$ or $p\pi$ respectively when t is a zero of order p. In describing the t-axis this comes to avoid the zeros of $p(t)$ by means of small semi-circles in the negative or positive sense respectively. In a discontinuity point t_0 we define $\arg^- p(t_0)$ or $\arg^+ p(t_0)$ as the mean value of the limits from the two sides of t_0.

Put now $a^+(t) = \arg^+ p(t+1) - \arg^+ p(t)$, i.e., $a^+(t)$ is the variation of $\arg^+ p(t)$ along a segment of length 1; similarly define $a^-(t)$. Our main result (Theorem I) is that *the function* $a^+(t)$, (respectively $a^-(t)$), *is almost periodic in the sense of Riemann-Stepanoff.* (The R. S. a. p. functions form a subclass of the S. a. p. functions; see [3] or below).

As a consequence of this theorem we prove (Theorem II) that $\arg^+ p(t)$ *may be written in the form*

$$\arg^+ p(t) = ct + \int_0^t \alpha(u)\,du + \psi(t),$$

where c *is a constant,* $\alpha(t)$ *a Bohr a. p. function with mean-value zero and* $\psi(t)$ *a (bounded) R. S. a. p. function.* The first result of this type is due to Wintner [6].

It is useless to try to prove that the remainder $\arg^+ p(t) - ct$ is bounded. For, as shown by Bernstein [1], even in the case $N = 3$, the remainder is generally unbounded.

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In our proofs we use a deep result given by Jessen and Tornehave in [5]; but instead of applying the Kronecker-Weyl theorem (as Jessen and Torenhave) or the ergodic theorem (as Wintner) we use the theory of Riemann-Stepanoff a. p. functions.

We add that our theorems may be proved in a more general setting: $p(t)$ may be replaced by a function $f(s)$, almost periodic in a strip, having a finite integral base and an analytic *spatial extension.*

Notations and definitions. Let E be a measurable set and let its characteristic function be $c(x)$. We define the *S-measure* of E as

$$S(E) = \sup_{-\infty<x<\infty} \int_x^{x+1} c(u)\,du.$$

The complement of E with respect to the set R of all real numbers will be denoted by E^*.

A function $f(x)$ is said to be *almost periodic in the sense of Riemann-Stepanoff* (R. S. a. p.) if to every $\delta > 0$ there corresponds two trigonometric polynomials $p(x)$, $q(x)$ such that $p(x) \leqq f(x) \leqq q(x)$ and $D_S(p, q) < \delta$. Here $D_S(p, q)$ is the usual Stepanoff distance between the two functions $p(x)$, $q(x)$.

In order that the bounded function $f(x)$ be R. S. a. p. it is necessary and sufficient that to every $\delta > 0$ there corresponds a measurable set E and numbers $k > 0$, $\mu_1, \cdots, \mu_m$ such that $S(E^*) < \delta$, and such that $|f(x) - f(x')| < \delta$ provided $x \in E$ and provided that $|\mu_n x - \mu_n x'| < k \pmod{2\pi}$, $(n = 1, \cdots, m)$. See [3].

Let the exponents $\lambda_1, \cdots, \lambda_N$ belong to the modul

$$M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$$

where the numbers $\mu_1, \cdots, \mu_m$ are linearly independent and the set of coefficients $h_1, \cdots, h_m$ runs through all sets of integers. Denoting the inner product $h_1x_1 + \cdots + h_mx_m$ of two vectors $h = (h_n)$ and $x = (x_n)$ by hx, and allowing terms with the coefficient 0 we may write $p(t)$ in the form $p(t) = \sum a_h \exp(ih\mu t)$. The function $P(x) = \sum a_h \exp(ihx)$ is called the *spatial extension* of $p(t)$. Clearly $P(\mu t) = p(t)$. Spatial extension may also be defined for any almost periodic function $f(t)$ with a finite integral base $\mu_1, \cdots, \mu_m$.

Proofs of the theorems. We shall need the following lemmas:

LEMMA I. *Let $D(x_1, \cdots, x_m) = D(x_n)$ be a function of $x_1, \cdots, x_m$, of period 2π in each x_n, analytic in the domain $|x_n - x_n^0| \leqq 3h$, $(n = 1,$*

$\cdots, m)$, *and not identically zero. Let* $\mu_1, \cdots, \mu_m$ *be linearly independent numbers. Denote by* $T(h)$ *the set of points* t *at which* $|\mu_n t - x_n^0| \leqq h \pmod{2\pi}$, $(n = 1, \cdots, m)$. *Then to every* $\delta > 0$ *we can associate an* $\epsilon > 0$ *such that the set* $T(h, \epsilon)$ *of the points* $t \in T(h)$ *at which* $|D(\mu_n t)| < \epsilon$ *has an S-measure less than* δ.

Proof. Let $|y_n - x_n^0| < h \pmod{2\pi}$, $(n = 1, \cdots, m)$. We first show that the function of θ, $D(\mu_n\theta + y_n)$, analytic in the domain $|\mu_n\theta| < 2h \pmod{2\pi}$, $(n = 1, \cdots, m)$, does not vanish identically. In fact, due to the linear independence of the μ_n, the points $(\mu_n\theta + y_n)$ are everywhere dense, mod 2π, in the set of points (x_n). Hence the set of points $(\mu_n\theta + y_n)$ for which $|\mu_n\theta + y_n - y_n| \leqq 2h \pmod{2\pi}$, $(n = 1, \cdots, m)$ is everywhere dense, mod 2π, in the set of points (x_n) for which $|x_n - y_n| < 2h$ $(n = 1, \cdots, m)$. Suppose therefore that the analytic function $D(\mu_n\theta + y_n)$ vanishes at every θ for which $|\mu_n\theta| \leqq 2h \pmod{2\pi}$, $(n = 1, \cdots, m)$, then the function $D(x_n)$ vanishes at every point (x_n) at which $|x_n - y_n| < 2h$ $(n = 1, \cdots, m)$, and hence everywhere, against the hypothesis. Thus $D(\mu_n\theta + y_n)$ does not vanish identically.

Suppose now that the lemma is false. Then there exists a $\delta > 0$, a sequence of points t_r and a sequence of positive numbers $\epsilon_r \to 0$ such that, for the Lebesgue measure μ:

$$\mu\{T(h, \epsilon_r) \cap (t_r - \tfrac{1}{2}, t_r + \tfrac{1}{2})\} \geqq \delta.$$

Hence, choosing an arbitrary $\theta_r \in T(h, \epsilon_r) \cap (t_r - \frac{1}{2}, t_n + \frac{1}{2})$ we have

$$\mu\{T(h, \epsilon_r) \cap (\theta_r - 1, \theta_r + 1)\} \geqq \delta.$$

Let $t \in T(h, \epsilon_r) \cap (\theta_r - 1, \theta_r + 1)$. Put $\theta = t - \theta_r$. Since

(1) $$|\mu_n\theta_r - x_n^0| \leqq h \pmod{2\pi}, \qquad |\mu_n t - x_n^0| \leqq h \pmod{2\pi}, \quad (n = 1, \cdots, m),$$

then θ belongs to the domain Θ defined by the inequalities $|\theta| \leqq 1$ and $|\mu_n\theta| \leqq 2h \pmod{2\pi}$, $(n = 1, \cdots, m)$. Moreover $|D(\mu_n\theta + \mu_n\theta_r)| < \epsilon_r$; Thus:

(2) the set of points $\theta \in \Theta$ for which $|D(\mu_n\theta + \mu_n\theta_r)| < \epsilon_r$ has a measure $\geqq \delta$.

Put

$$x_n^r = \mu_n\theta_r, \mod 2\pi, \qquad 0 \leqq x_n^r < 2\pi, \quad (n = 1, \cdots, m),$$

and let $y = (y_n)$ be a limiting point of the (x_n^r). By (1), $|y_n - x_n^0| \leqq h \pmod{2\pi}$, $(n = 1, \cdots, m)$. Now the function of θ, $D(\mu_n\theta + y_n)$ does not

vanish identically. It is analytic in the domain Θ and this domain consists of a finite number of intervals. Hence, we can find an $\epsilon > 0$ such that

(2′) the set of points $\theta \in \Theta$ for which $|D(\mu_n\theta + y_n)| < \epsilon$ has a measure $< \delta$.

On the other hand, we can find an r such that $\epsilon_r < \epsilon/2$ and such that

$$|D(\mu_n\theta + \mu_n\theta_r) - D(\mu_n\theta + y_n)| < \epsilon/2,$$

for $\theta \in \Theta$. If $\theta \in \Theta$ is such that $|D(\mu_n\theta + \theta_r\mu_n)| < \epsilon_r$, then $|D(\mu_n\theta + y_n)| < \epsilon$, so that, by (2), the set of points $\theta \in \Theta$ at which $|D(\mu_n\theta + y_n) < \epsilon$ has a measure $\geqq \delta$: but this contradicts (2′), and the hypothesis that the lemma is false is untenable.

COROLLARY. *Let* $p(t) = \sum_1^N a_n \exp(i\lambda_n t)$ *be a trigonometric polynomial. Then, to every* $\delta > 0$ *we can associate an* $\epsilon > 0$ *such that the set of points* t *at which* $|p(t)| < \epsilon$ *has an S-measure* $< \delta$.

We just apply the lemma to the analytic spatial extension

$$P(x_n) = P(x) = \sum a_h \exp(ihx).$$

Remark. We shall use this corollary in the proof of Theorems I and II. To prove these theorems, not for a polynomial $p(t)$, but for a function $f(s)$, almost periodic in a strip and having a finite integral base and an analytic spatial extension, one should replace the corollary by the following statement whose direct proof presents no difficulty: "To every σ in the strip and to every $\delta > 0$ we can associate an $\epsilon > 0$ such that the set of points t at which $|f(\sigma + it)| < \epsilon$ has an S-measure less than δ."

THEOREM I. *Let* $p(t) = \sum_1^N a_n \exp(i\lambda_n t)$ *be a trigonometric polynomial. If* $a^+(t) = \arg^+ p(t+1) - \arg^+ p(t)$, *then* $a^+(t)$ *is an R.S.a.p. function.*

Proof. Using the notations above we put

$$P(x,t) = \sum a_h \exp(ihx) \exp(ih\mu t).$$

Denote by $a^+(x_n, 0)$ or $a^+(x, 0)$ the variation of $\arg^+ P(x,t)$ on the segment $(0,1)$ of the t-axis (so that $a^+(\mu_n t, 0) = a^+(t)$). Let $z(x,0)$ be the number of zeros of $P(x,t)$ on the same segment, a zero at an end point being counted with only half its multiplicity. We use the following result proved by Jessen and Tornehave ([5], p. 264):

"To every x^0 we can associate a neighbourhood $I(x^0, 3h^0)$: $|x_n - x_n{}^0| \leqq 3h^0$ $(n = 1, \cdots, m)$, and a finite number of functions not identically

zero $D_1(x_1, \cdots, x_m) = D_1(x_n), \cdots, D_l(x_n)$, of period 2π in each x_n, analytic in the domain $I(x^0, 3h^0)$ and such that $z(x, 0)$ is a continuous function of x at every point $x \in I(x^0, 3h^0)$ at which $D_1(x_n) \neq 0, \cdots, D_l(x_n) \neq 0$."

Replacing the functions $D_1(x_n), \cdots, D_l(x_n)$ by their product $D^0(x_n)$, we see that $z(x, 0)$ is a continuous function of x at every point $x \in I(x^0, 3h^0)$ at which $D^0(x_n) \neq 0$.

It is however obvious that for a point x at which $P(x, 0) \neq 0$ and $P(x, 1) \neq 0$ the continuity of $z(x, 0)$ implies the continuity of $a^+(x, 0)$.

The points x for which $0 \leqq x_n \leqq 2\pi$ $(n = 1, \cdots, m)$ form a compact set K. The covering of K by the sets $I(x^0, h^0)$ contains a finite covering, say, $I(x^0, h^0), \cdots, I(x^r, h^r)$. Let $\epsilon > 0$ be arbitrary; let $I(x^j, h^j, \epsilon)$ be the set of points of $I(x^j, h^j)$ at which $|D^j(x_n)| \geqq \epsilon$ and let P_ϵ be the set of x for which $|P(x, 0)| \geqq \epsilon$, $|P(x, 1)| \geqq \epsilon$. Then $a^+(x, 0)$ is continuous at every point of the set

$$K_\epsilon = P_\epsilon \cap \bigcup_1^r I(x^j, h^j, \epsilon).$$

K_ϵ being compact, we prove in a standard manner that to every $\delta > 0$ we can associate an $h = h_\delta$ such that $|a^+(x, 0) - a^+(x', 0)| < \delta$, provided $x \in K_\epsilon$ and provided $|x_n - x_n'| < h \pmod{2\pi}$, $(n = 1, \cdots, m)$.

We now go back to the variable t and put $x_n = \mu_n t \pmod{2\pi}$ $0 \leqq x_n \leqq 2\pi$, $(n = 1, \cdots, m)$. Let T_ϵ be the set of points t for which $x \in K_\epsilon$. Then $|a^+(t) - a^+(t')| < \delta$, provided $t \in T_\epsilon$ and provided $|\mu_n t - \mu_n t'| < h \pmod{2\pi}$, $(n = 1, \cdots, m)$. Also the function $a^+(t)$ is bounded (see e.g., [5], p. 179).

To prove that $a^+(t)$ is an R. S. a. p. function there remains to show that we can find an $\epsilon > 0$ for which $S(T_\epsilon^*) < \delta$.

Now if $t \in T_\epsilon^*$ then either $x \in P_\epsilon^*$ or for some $j = 1, \cdots, r$, $x \in I(x^j, h^j)$, $x \notin I(x^j, h^j, \epsilon)$, i.e., t belongs to the set of points t for which $|D^j(\mu_n t)| < \epsilon$, and $|\mu_n t - x_n^j| < h^j \pmod{2\pi}$, $(n = 1, \cdots, m)$. By Lemma I, we can always choose $\epsilon > 0$ such that each of these r sets has an S-measure less than $\delta/(r+1)$. By the Corollary, the set of points t for which $x \in P_\epsilon^*$ may also be given an S-measure less than $\delta/(r+1)$. This gives $S(T_\epsilon^*) < \delta$, and the theorem is proved.

Remark. In the same way we prove that for any $\delta > 0$ the function $[\arg^+ p(t+\delta) - \arg^+ p(t)]$ is an R. S. a. p. function.

LEMMA II. *If $b(t)$ is an S. a. p. function and if $B(t) = \int_0^t b(u)\,du$ is bounded, then $B(t)$ is a Bohr a. p. function.*

The proof is identical with the classical proof of Bohr's theorem stating

that if an indefinite integral of a Bohr a. p. function is bounded, then it is also a Bohr a. p. function (see e. g., [2], p. 7).

LEMMA III. *Let* $p(t) = \sum_1^N a_n \exp(i\lambda_n t)$ *be a trigonometric polynomial. Then there exists a Bohr a. p. function* $A(t)$ *such that*

$$\arg^+ p(t) = \int_0^t A(u)\,du + O(1).$$

Proof. We have

$$(3) \qquad \int_0^n a^+(t)\,dt = \int_n^{n+1} \arg^+ p(t)\,dt - \int_0^1 \arg^+ p(t)\,dt = \arg^+ p(n) + O(1),$$

since the variation of $\arg^+ p(t)$ along any segment of length 1 is bounded. Put now

$$A(t) = \int_t^{t+1} a^+(u)\,du = \int_0^1 a^+(t+u)\,du.$$

Since $a^+(u)$ is an S. a. p. function then $A(t)$ is a Bohr a. p. function. But we have

$$\int_0^n A(t)\,dt = \int_0^n \int_0^1 a^+(t+u)\,du\,dt = \int_0^1 \left(\int_0^n a^+(t+u)\,dt\right) du$$

$$= \int_0^1 \phi_n(u)\,du \quad (\text{say}) = \phi_n(\theta) \qquad (0 \leqq \theta \leqq 1)$$

$$= \int_0^n a^+(\theta+t)\,dt = \int_\theta^{\theta+n} a^+(t)\,dt. \quad \text{Hence}$$

$$\int_0^n A(t)\,dt - \int_0^n a^+(t)\,dt = \int_n^{n+\theta} a^+(t)\,dt - \int_0^\theta a^+(t)\,dt = O(1).$$

This gives, by (3), $\arg^+ p(n) = \int_0^n A(t)\,dt + O(1)$, and finally

$$\arg^+ p(t) = \int_0^t A(u)\,du + O(1).$$

THEOREM II. *Let* $p(t) = \sum_1^N a_n \exp(i\lambda_n t)$ *be a trigonometric polynomial. Then*

$$\arg^+ p(t) = ct + \int_0^t \alpha(u)\,du + \psi(t),$$

where c *is a constant,* $\alpha(t)$ *a Bohr a. p. function with mean value zero and* $\psi(t)$ *a (bounded) R. S. a. p. function.*

Proof. If c is the mean value of $A(t)$, then, by Lemma III, we can write

$$\arg^+ p(t) = ct + \int_0^t \alpha(u)\,du + \psi(t),$$

where $\alpha(t)$ is a Bohr a. p. function with mean-value zero and $\psi(t)$ is bounded. We shall prove that $\psi(t)$ is an R. S. a. p. function. Put

$$B(t) = \int_0^t \{\psi(u+1) - \psi(u)\}\,du = \int_t^{t+1} \psi(u+1)\,du - \int_0^1 \psi(u)\,du = O(1).$$

Since $\psi(t+1) - \psi(t)$ is an S. a. p. function, we see, by Lemma II, that $B(t)$ is a Bohr a. p. function, so that the function

$$\psi_1(t) = \int_t^{t+1} \psi(u)\,du$$

is also a Bohr a. p. function.

In the same way, using the fact that $\arg^+ p(t+\gamma) - \arg^+ p(t)$ is an R. S. a. p. function we prove that

$$\psi_\gamma(t) = \gamma^{-1} \int_t^{t+\gamma} \psi(u)\,du$$

is a Bohr a. p. function for any $\gamma > 0$.

Let now $\delta > 0$ be arbitrary. By the Corollary of Lemma I we can find an $\epsilon > 0$ such that, for the set $E_{2\epsilon}$ of points t at which $|p(t)| \geqq 2\epsilon$, we have

$$S(E_{2\epsilon}^*) < \delta. \tag{4}$$

Now ϵ being fixed, we can find an $h_0 > 0$ such that

$$|\arg^+ p(t+h) - \arg^+ p(t)| < \delta/6,$$

provided $t \in T_\epsilon$ and $|h| < h_0$. But

$$\psi(t+h) - \psi(t) = -ch - \int_t^{t+h} \alpha(u)\,du + \arg^+ p(t+h) - \arg^+ p(t).$$

Therefore we may find an $h_1 > 0$ such that $|\psi(t+h) - \psi(t)| < \delta/3$ for $t \in T_\epsilon$ and $|h| < h_1$. This shows that

$$|\psi_\gamma(t) - \psi(t)| < \delta/3 \tag{5}$$

for $t \in T_\epsilon$ and $|\gamma| < h_1$.

Take now a fixed positive $\gamma < h_1$. Since $\psi_\gamma(t)$ is a Bohr a. p. function we can find an $h_2 > 0$ and numbers $\nu_1, \cdots, \nu_{m'}$ such that

$$|\psi_\gamma(t+\tau) - \psi_\gamma(t)| < \delta/3, \tag{6}$$

if only $|\nu_n\tau| < h_2 \pmod{2\pi}$ $(n=1,\cdots,m')$. Also we can find an $h_3>0$ such that

$$|p(t+\tau)-p(t)|<\epsilon, \tag{7}$$

provided that $|\mu_n\tau| < h_3 \pmod{2\pi}$, $(n=1,\cdots,m)$. Now let $t\in E_{2\epsilon}\subset E_\epsilon$ and let

$$|\nu_n\tau| < h_2, \bmod 2\pi, \ (n=1,\cdots,m'); \ |\mu_n\tau| < h_3, \bmod 2\pi, \ (n=1,\cdots,m).$$

Then, by (7), $|p(t+\tau)|\geqq\epsilon$, so that $t+\tau\in T_\epsilon$. Therefore, by (5),

$$|\psi(t+\tau)-\psi_\gamma(t+\tau))<\delta/3. \tag{8}$$

Relations (8), (6) and (5) now give $|\psi(t+\tau)-\psi(t)|<\delta$. This shows, by (4), that $\psi(t)$ is an R. S. a. p. function and completes the proof of Theorem II.

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ON DIFFERENTIAL EQUATIONS ON THE TORUS.*

By SHLOMO STERNBERG.

1. The study of differential equations on the torus was initiated by Poincaré in [3]. It was conjectured there, that for non-singular systems under suitable smoothness assumptions, either there exists a periodic solution or every solution path is dense on the torus. This result was first proved by Denjoy [1], cf. also Siegel [5]. More recently, it was shown by Saito [4] that if the flow generated by the differential equations is assumed to be measure preserving, then either every solution is periodic or the flow is ergodic, depending on the rationality or irrationality of a certain ratio. Thus all measure preserving flows on the torus behave qualitatively like a rectilinear flow on the Euclidean torus. We shall improve upon this result by showing that any flow on the torus with an invariant integral actually is rectilinear in the sense that by suitable choice of coordinates on the torus one can arrange that all the trajectories are straight lines. However the linearization of the time dependence depends intimately upon the arithmetical properties of the flow and becomes an issue of small divisors. It may be remarked that since Hamiltonian flows have an invariant integral, the following results are valid for a Hamiltonian system on the torus.[1]

2. We now state our principal theorem.

THEOREM 1. *Let the system of differential equations*

$$dx/dt = f(x,y), \qquad dy/dt = g(x,y) \tag{1}$$

be of class C^n and defined on the torus, i.e. let f and g be periodic of period one in both variables. Furthermore assume that the differential equations (1) *possess an integral invariant $U(x,y)$ of class C^n defined on the torus. Then there exists a change of coordinates*

$$X = X(x,y), \qquad Y = Y(x,y) \tag{2}$$

* Received October 4, 1956.

[1] The results of this paper were obtained during participation in a seminar conducted by Professor Wintner.

such that in the new coordinate system the differential equations (1) *have the form*

$$dX/dt = F(X,Y), \qquad dY/dt = \gamma F(X,y). \tag{3}$$

Thus in these new coordinates the trajectories are all straight lines.

In the proof of this theorem, essential use will be made of the non-bounding cycle without contact introduced by Siegel in [5]. We remark that in the present case, this cycle can be chosen to be of class C^{n+1}. This follows immediately from Siegel's construction. In fact, the cycle without contact is constructed by considering the orthogonal trajectories to (1). If the orthogonal system has a periodic solution, this provides the desired cycle. It is clearly of class C^{n+1} since the differential equations are of class C^n. In the complementary case one closes up an orthogonal trajectory in the neighborhood of a limit point where the vector field is approximately constant; and this can be done in a smooth manner. For the details the reader is referred to [5].

3. Let Γ be the cycle without contact. We wish to construct a family of such cycles filling up the torus. The natural thing to do, would be to allow the points of Γ to follow the flow generated by the differential equations. However, this procedure, as it stands, would lead to difficulties since the points flow with different speeds along different trajectories, thus leading to the situation where one point on Γ will have returned to Γ at a given time, whereas another point might not have. This difficulty can be avoided simply by speeding up the flow on the slow trajectories and slowing down the flow on the fast ones: if we denote by $t(s)$ the length of time it takes for a point on the cycle (parametrized by s) to return to Γ under the flow, then $t(s)$ is a function of class C^{n+1}. Denote by $\Gamma(s;\tau)$ the image of the point $\Gamma(s)$ under the flow corresponding to the time $t = t(s)\cdot\tau$. The family of curves $\Gamma(\cdot\,;\tau)$ provides us with the desired family of cycles without contact. We can now introduce the function $\tau(x,y)$ which tells what cycle $\Gamma(\cdot\,;\tau)$ the point (x,y) is on. This function essentially tells us how far the point (x,y) is along a trajectory; we now wish to introduce a parameter which tells us which trajectory the point (x,y) is on. For this purpose we make use of the integral invariant. Now the fact that U is an integral invariant can be expressed analytically as

$$(Uf)_x + (Ug)_y = 0. \tag{4}$$

This, in turn, implies the existence of a function $H(x, y)$ satisfying

$$H_y = Uf, \qquad H_x = -Ug. \tag{5}$$

Thus

$$dH/dt = H_x f + H_y g = 0. \tag{6}$$

i.e., H is an integral of the flow. The function H is not defined on the torus but in the plane, i.e., it is not necessarily periodic. However, by virtue of (5), the partial derivatives of H are periodic and we may write

$$H(x, y) = ax + by + h(x, y), \tag{7}$$

where h is a function of class C^{n+1} on the torus. Since the system (1) is non-singular, the function H cannot assume the same value for two distinct trajectories in the plane. We now set

$$X = \tau(x, y), \qquad Y = \alpha\tau(x, y) + \beta H(x, y), \tag{8}$$

where α and β are chosen so that

$$\begin{aligned} \alpha(\tau(x, y) - \tau(x + 1, y)) + \beta(H(x, y) - H(x + 1, y) &= m, \\ \alpha(\tau(x, y) - \tau(x, y + 1)) + \beta(H(x, y) - H(x, y + 1) &= n, \end{aligned} \tag{9}$$

where m and n are integers. It is clear that the change of coordinates (8) transforms (1) to the form (3) with $\gamma = \alpha$.

4. In order to obtain an explicit formula for γ we first show that by proper linear area preserving changes of coordinates we can arrange that the cycles without contact are homotopic to the cycle $x = 0$. This is a well known result that we include here for completeness.

LEMMA 1. *Let K be a non-bounding non-self-intersecting cycle defined on the torus T. Then there exists a linear area preserving homeomorphism of T onto a torus T' such that image K' of K is homotopic to the cyclic $x = 0$.*[2]

It is clear from the structure of the fundamental group of the torus that we can find a linear automorphism of the torus of the form $x' = ax + by$, $y' = cx + dy$ with integral a, b, c, d and $ad - bc = \pm 1$ and is therefore area preserving such that K is homotopic to some multiple m of the cycle $x = 0$. We now show that $m = 1$. In fact, let $\boldsymbol{K}$ denote the curve in the plane which covers K. If $m = \pm 1$ then the translate of $\boldsymbol{K}$ by the vector

[2] The following proof of the lemma was obtained jointly with Harry Furstenberg.

$(0,1)$ does not intersect $\boldsymbol{K}$ and so must lie entirely in one of the two domains in the plane bounded by $\boldsymbol{K}$. Repeating this procedure m times leads to a contradiction.

In these new coordinates the function τ satisfies the equations

$$\tau(x+1,y)-\tau(x,y)=1, \qquad \tau(x,y+1)-\tau(x,y)=0;$$

we may thus choose α and β in (10) as $-a/b$ and $1/b$ respectively. Here it is to be remarked that a, b, f and g are all to be caiculated in the new coordinate system (and that f and g transform as vectors).

By (5) and (7) we have

$$(10) \qquad a=H(x+1,y)-H(x,y)=\int_0^1 H_x(x,y)\,dx$$
$$=\int_0^1\int_0^1 H_x(x,y)\,dxdy=-\iint_T U g\,dxdy.$$

Similarly,

$$(11) \qquad b=\iint_T U f\,dxdy.$$

Combining (10), (11) and (9), we obtain

$$(12) \qquad \gamma=\iint_T U f\,dxdy \Big/ \iint_T U g\,dxdy.$$

This formula was obtained in the new coordinate system. However since both sides of (12) transform in the same manner under linear change of coordinates, (12) holds in the original coordinate system.

5. The results of paragraph 2 may be summed up by saying that given any system (2) with a smooth invariant measure, we can find a smooth change of coordinates which makes all the trajectories of (2) linear. In these new coordinates the dependence on the time, is still, however, non-linear. The question of whether or not we can linearize the time dependence, depends on the arithmetic nature of the invariant τ. The following result is due to Kolmorgoroff [2]. It is essentially equivalent to a previous result of Wintner [6].

THEOREM. *Let $F(X,Y)$ be an analytic function on the torus. A sufficient condition for the existence of an analytic change of coordinates sending the system* (3) *into the system*

$$(13) \qquad dx/dt=A, \qquad dy/dt=B; \qquad A/B=\gamma.$$

is that there exists two constants $K>0$ and $h>0$, such that for all integers m and n,

$$|m-n\gamma|>Kh^n. \tag{14}$$

Remark. If condition (14) is not satisfied, then there exists a γ and an F such that (3) can be transformed into the form (13) by a change of coordinates which is of class C^∞ or of class C^k but not C^{k+1} for any k.

It is clear that any two coordinate systems in which (3) has the form (14) can differ only by a linear transformation, so that as far as smoothness considerations are concerned, the change of coordinates is essentially unique. Now if we were operating in the plane, the change of coordinates

$$x^*=kT(X,Y)=k\int_0^X(1/F(z,Y+\gamma(z-x)))dz,$$

$$y^*=Y+\gamma(x^*-X),$$

would have the desired effect. In fact all we have to check is dx^*/dt. But

$$\partial x^*/\partial X=k(1/F(X,Y)+k\int_0^X(1/F)_Y dz\cdot(-\gamma),\quad \partial x^*/\partial Y=k\int_0^X(1/F)_Y dz,$$

so that $dx^*/dt=F(X,Y)\partial x^*/\partial X+\gamma F(X,Y)\partial x^*/\partial Y=k$.

We now have to modify x^* so that it becomes periodic. We may do this without affecting the form (13) by adding a suitable function of $Y+\gamma X$. Setting $G=1/F$ we have (since G is periodic),

$$T(X+1,Y)=\int_0^{X+1}G(z,Y+\gamma(z-X))dz=\int_{-1}^{X}G(z,Y-\gamma X+\gamma(z-1))ds.$$

Thus $$T(X+1,Y)-T(X,Y)=\int_0^1 G(s,Y+\gamma(s-X))ds-S(Y-\gamma X),$$

where

$$S(Z)=\int_0^1 G(z,Z+\gamma(s-1))\,dz=T(1,Z).$$

Setting $1/k=\int_0^1 S(Z)dZ=\int_0^1\int_0^1(1/F)dxdy$, we see that if $R(Z)$ is a solution of the functional equation

$$R(s+\gamma)-R(s)=S(s)-1/k, \tag{15}$$

then the change of coordinates

$$x=k(T(X,Y)+R(Y-X),\qquad y=Y+\gamma(x-X), \tag{16}$$

has the desired properties. But (15) is a difference equation in angular variables so that the results of [6] are applicable.

Expanding S into a Fourier series

$$S(w) = 1/k + \sum_{n \neq 0} a_n e^{2\pi i n w},$$

equation (15) has the formal solution

$$R(w) = \sum_{n \neq 0} a_n e^{2\pi i n w} / (e^{2\pi i n \gamma} - 1).$$

Since $S(w)$ is analytic, condition (14) assures the analyticity of R. As we can choose S arbitrarily by proper choice of F, this proves the remark following the theorem.

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ON THE IMBEDDING OF V-MANIFOLDS IN PROJECTIVE SPACE.*[1]

By W. L. Baily.

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1. Introduction. In a previous paper [2], we defined what is meant by a V-bundle over a V-manifold $\mathcal{V}$ and proved analogs of the usual regularity and decomposition theorems, which, in fact, contain the usual theorems as special cases. In concluding that paper, we remarked that Kodaira's results for Hodge varieties [15], of which a generalization was announced in [1], could be extended to V-manifolds. In this paper, we prove an analog of Dolbeault-Kodaira's theorem [13] and use this together with properties of the quadratic transformation to prove our main result which states that, if $\mathcal{V}$ is a compact, complex analytic V-manifold, and if there exists a complex line bundle over $\mathcal{V}$ whose characteristic class contains a closed positive definite form of type $(1,1)$, then $\mathcal{V}$ is an algebraic variety. In terms to be defined later, there then exists a biregular complex analytic mapping Φ of $\mathcal{V}$ into some complex projective space $\mathfrak{S}^m$ such that Φ is a homeomorphism of $\mathcal{V}$ onto some algebraic variety, and Φ induces an isomorphism of the ring of local holomorphic functions at $x \in \mathcal{V}$ with the local ring of $\Phi(\mathcal{V})$ at $\Phi(x)$.

Examples readily dealt with by our result are the following two: (1) If $\mathcal{D}$ is a bounded domain in C^n and G a discontinuous group of complex analytic transformations of $\mathcal{D}$ such that $\mathcal{D}/G$ is compact, then $\mathcal{D}/G$ is an algebraic variety, even when non-trivial elements of G have fixed points in $\mathcal{D}$. (2) If X is a non-singular algebraic variety and if G is a finite group of complex analytic transformations of X, then X/G is an algebraic variety.

It also turns out that an example of a V-manifold recently given by Satake, in connection with Siegel's modular functions [18], is algebraic. The proof is somewhat more difficult, depending on the construction of a complex line bundle whose characteristic class contains a positive definite form, and will be given in a later paper.

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2. DeRham's theorem and Dolbeault's theorem. We let $\mathcal{V}$ be a C^∞ V-manifold. If B is a V-bundle over $\mathcal{V}$ with a vector space as fibre, it is clear that the sections of B over an open subset $\mathcal{O}$ of $\mathcal{V}$ form an additive group. We let A^p denote the bundle of differential p-forms for $\mathcal{V}$ and let $\mathcal{F}^p$ denote the sheaf of germs of C^∞ sections of A^p. It is clear that $\mathcal{F}^p$ is a fine sheaf, since there exists a C^∞ partition of unity subordinate to any locally finite covering of $\mathcal{V}$, and therefore (see [21]) $H^q(\mathcal{V}, \mathcal{F}^p) = 0$ if $q \geqq 1$, while, as always, $H^0(\mathcal{V}, \mathcal{F}^p) = \mathfrak{M}^p$ is the module of C^∞ sections of A^p. We let D^p denote the subsheaf of $\mathcal{F}^p$ consisting of germs of forms ϕ such that $d\phi = 0$ (i.e., such that ϕ is closed). Then since $d^2 = 0$, d induces a homomorphism, also denoted by d, of $\mathcal{F}^p$ into D^{p+1}. We now show that d maps $\mathcal{F}^p$ onto D^{p+1}. Let $x \in \mathcal{V}$, let ϕ be a closed C^∞ differential $(p+1)$-form (i.e., a C^∞ section of A^{p+1}) in a neighborhood W of x and let $\{U, G, v\}$[2] be a l.u.s. for $\mathcal{V}$ such that $x \in \bar{U} \subset W$. Then in U, ϕ is represented by a C^∞ G-invariant differential $(p+1)$-form ϕ_U such that $d\phi_U = 0$. Let $y \in v^{-1}(x)$. By the Poincaré lemma there exists a C^∞ differential p-form ψ in a small neighborhood N of y such that $d\psi = \phi_U$, and we may assume that $gN \cap N$ is empty unless g belongs to the subgroup G_y of G leaving y fixed, while $gN = N$ if $g \in G_y$. We put $\psi_N{}^* = (1/\text{ord}\, G_y) \sum_{g \in G_y} \psi_g$, where $\psi_g(\eta) = g^*\psi(g\eta)$, $\eta \in N$, and g^* is the mapping of differential forms dual to g. Then since d commutes with the mapping of differential forms dual to a differentiable mapping of one manifold into another, it follows that $d\psi_N{}^* = \phi_U$. Since $\psi_N{}^*$ is invariant under G_y, $\psi_N{}^*$ induces a C^∞ differential p-form ψ^* in a small neighborhood of x on $\mathcal{V}$ such that $d\psi^* = \phi$. Hence $d: \mathcal{F}^p \to D^{p+1}$ is onto. Therefore the sequence of sheaves

$$0 \to D^p \to \mathcal{F}^p \to D^{p+1} \to 0$$

is exact. From the associated exact cohomology sequence we obtain by a well-known inductive process, using the fact that $H^q(\mathcal{V}, \mathcal{F}^p) = 0$ if $q \geqq 1$, that $H^q(\mathcal{V}, D^0) \cong \mathfrak{Z}^q/d\mathfrak{M}^{q-1}$, $q \geqq 1$, where $\mathfrak{Z}$ is the module of C^∞ closed forms of degree q, and since D^0 is just the constant sheaf of real numbers, $H^q(\mathcal{V}, R) \cong \mathfrak{Z}^q/d\mathfrak{M}^{q-1}$; thus we have obtained another proof (see Satake [17]) of de Rham's theorem which is analogous to that given by Weil for C^∞ manifolds [24]. Let $\mathcal{V}$ be supplied with a Riemann metric. Coupling the above result with Theorem H of [2] we have $H^q(\mathcal{V}, R) = \mathfrak{H}^q$, where $\mathfrak{H}^q$ is the finite dimensional module of real harmonic differential forms of degree q on $\mathcal{V}$.

[2] In this paper we denote a l.u.s. by $\{U, G, v\}$ instead of by $\{U, G, \phi\}$ as in [2]. Furthermore, ∂ and $\bar{\partial}$ of [2] have been replaced by d' and d'' respectively.

Now let $\mathcal{V}$ be a compact, complex analytic V-manifold and let B be a complex line bundle over $\mathcal{V}$. We let $A^{r,s}$ denote the bundle of forms of type (r,s) on $\mathcal{V}$. Then we denote by $\mathcal{F}^{r,s}(B)$ the sheaf of germs of C^∞ sections of $B\otimes A^{r,s}$. It is clear that $\mathcal{F}^{r,s}(B)$ is a fine sheaf so that $H^q(\mathcal{V},\mathcal{F}^{r,s}(B))=0$ if $q\geqq 1$, while, if $\mathfrak{M}(B)^{r,s}$ is the module of C^∞ forms of type (r,s) with coefficients in B, $H^0(\mathcal{V},\mathcal{F}^{r,s}(B))=\mathfrak{M}(B)^{r,s}$. We let $D^{r,s}(B)$ denote the subsheaf of $\mathcal{F}^{r,s}(B)$ consisting of germs of C^∞ sections ϕ of $B\otimes A^{r,s}$ such that $d''\phi=0$. Since $d''^2=0$, it follows that d'' induces a homomorphism of $\mathcal{F}^{r,s}(B)$ into $D^{r,s+1}(B)$. We may then use Dolbeault's lemma (see [11]) in the same role as we previously used Poincaré's lemma, together with the fact that d'' commutes with complex analytic mappings and multiplication by a holomorphic function, to prove, by use of an obvious local averaging process, that $d'':\mathcal{F}^{r,s}(B)\to D^{r,s+1}(B)$ is onto. Therefore the sequence of sheaves

$$0\to D^{r,s}(B)\to \mathcal{F}^{r,s}(B)\to D^{r,s+1}(B)\to 0$$

is exact. From the associated exact cohomology sequence we again obtain, by an inductive process, that

$$H^s(\mathcal{V},D^{r,0}(B))=\mathcal{Z}^{r,s}(B)/d''\mathfrak{M}^{r,s-1}(B),\quad s\geqq 1,$$

where $\mathcal{Z}^{r,s}(B)$ is the module of d''-closed forms of type (r,s) with coefficients in B; and since $D^{r,0}(B)$ is just the sheaf of germs of holomorphic r-forms with coefficients in B, which we denote by $\Omega^r(B)$, we obtain

$$H^s(\mathcal{V},\Omega^r(B))=\mathcal{Z}^{r,s}(B)/d''\mathfrak{M}^{r,s-1}(B),\quad s\geqq 1,$$

which is Dolbeault's theorem. If $\mathcal{V}$ is supplied with a Hermitian metric and B is supplied with a C^∞ metric, we may couple the above result with Theorem K of [2] to obtain

THEOREM D-K. $H^s(\mathcal{V},\Omega^r(B))=\mathcal{H}(B)^{r,s}$,

which we will refer to as Dolbeault-Kodaira's theorem.

3. Certain definitions for V-manifolds. Let $\mathcal{V}$ be a compact C^∞ V-manifold. Let $\{U,G,v\}$ be a l.u.s. for $\mathcal{V}$, let $x\in U$, and let G_x be the subgroup of G leaving x fixed. We say that $x^*=v(x)\in\tilde{U}$ is a regular point on $\mathcal{V}$ if G_x consists of the identity alone; otherwise we say that x^* is singular. We let $m_x=\operatorname{ord}G_x$ and call m_x the order of x. We define $m_0=\text{l.c.m.}_{x^*\in\mathcal{V}}\,m_x$ and call m_0 the order of $\mathcal{V}$; clearly m_0 is finite since $\mathcal{V}$ is compact. It is evident that the above definitions do not depend on the

choice of l. u. s. The regular points of $\mathfrak{V}$ form a C^∞ manifold in the ordinary sense. From our definition in [2], it is clear that we may naturally identify the ring of germs of C^∞ functions at x^* on $\mathfrak{V}$ with the ring of germs of G-invariant C^∞ functions on U at x. If $\mathfrak{V}$ is a complex analytic V-manifold, it follows from a known theorem on removable singularities that the ring of germs of holomorphic functions at x^*, in addition to being naturally identified with the ring of germs of G-invariant holomorphic functions at x, also coincides with the ring of germs of continuous functions at x^* which are holomorphic at all the regular points of $\mathfrak{V}$ in a neighborhood of x.

We now let $\mathfrak{V}$ be a compact, complex analytic V-manifold. By a known theorem [6], we will assume, without loss of generality, that for each l. u. s. $\{U, G, v\}$ G acts as a finite linear group in the coordinates of U. By a divisor over $\mathfrak{V}$ we mean that, for each l. u. s. $\{U, G, v\}$, we are given a divisor $\boldsymbol{D}_U$ on U such that, if f belongs to the module $\boldsymbol{D}_U(gx)$ over the ring $\mathfrak{O}_{gx}$ of local holomorphic functions at $gx \in U$ which are locally multiples of $\boldsymbol{D}_U$, then $f \circ g$ belongs to $\boldsymbol{D}_U(x)$ for each $x \in U$, $g \in G$, and such that, if λ is an injection of $\{U', G', v'\}$ into $\{U, G, v\}$ and if f belongs to $\boldsymbol{D}_U(\lambda(y))$ at $\lambda(y) \in \lambda(U')$, then $f \circ \lambda$ belongs to $\boldsymbol{D}_{U'}(y)$. We say that $\boldsymbol{D}$ is an absolute divisor (i. e., locally linearly equivalent to zero) on $\mathfrak{V}$ if at each $x \in \mathfrak{V}$ there exists a l. u. s. $\{U, G, v\}$, $x \in \bar{U}$, such that $\boldsymbol{D}_U$ is the divisor (f_U) of a single meromorphic function f_U in U which can be expressed as the quotient of two relatively prime G-invariant holomorphic functions. It is not necessarily true that a divisor $\boldsymbol{D}$ over $\mathfrak{V}$ is an absolute divisor, but if m_0 is the degree of $\mathfrak{V}$, then $m_0\boldsymbol{D}$ is absolute. To see this, we let $x \in \mathfrak{V}$, let $\{U, G, v\}$ be a l. u. s., $x \in \bar{U}$, such that $\boldsymbol{D}_U$ is the divisor (f_U) of a quotient of two holomorphic functions, and let $m = \operatorname{ord} G$; f_U is not necessarily the quotient of two relatively prime, G-invariant, holomorphic functions, but, the product $\boldsymbol{p}_U$ of its tranlates under G clearly is, and $(\boldsymbol{p}_U^{(m_0/m)}) = m_0\boldsymbol{D}_U$ in view of the invariance of $\boldsymbol{D}_U$ under G. (In this connection see [20], exposé XX.)

Let B be a V-bundle over $\mathfrak{V}$ (see [2]). We may assume without loss of generality that B_U is the product bundle for each l. u. s. $\{U, G, v\}$. Then if $g \in G$ and $(z, \zeta) \in B_U$, $h_U(g)(z, \zeta) = (g^{-1}z, \eta_U(g)(z) \cdot \zeta)$, while if λ is an injection of $\{U, G, v\}$ into $\{U', G', v'\}$, $z \in U$, and $(\lambda(z), \zeta) \in B_{U'} \mid \lambda(U)$, then $\lambda^*(\lambda(z), \zeta) = (z, \xi_\lambda(\lambda(z)) \cdot \zeta)$, where $\eta_U(g)$ and ξ_λ are, respectively, mappings of U and $\lambda(U)$ into the group of the bundle; in particular, if B is a complex line bundle, $\eta_U(g)$ and ξ_λ are non-zero holomorphic functions. It is clear, from the definition of V-bundle, that $\eta_U(g_2)(g_1^{-1}z)\eta_U(g_1)(z) = \eta_U(g_1g_2)(z)$ and $\eta_U(g)(z)\xi_\lambda(\lambda(z)) = \xi_\lambda(\lambda(g^{-1}z))\eta_{U'}(g')(\lambda(z))$, where $g, g_1, g_2 \in G$ and $z \in U$, while if λ' is an injection of $\{U', G', v'\}$ into $\{U'', G'', v''\}$,

then $\xi_\lambda(\lambda(z))\xi_{\lambda'}(\lambda'\lambda(z)) = \xi_{\lambda'\lambda}(\lambda'\lambda(z))$. Conversely, if $\{\eta_U(g)\}$ and $\{\xi_\lambda\}$ are given subject to these conditions, it is clear that we obtain a V-bundle in a natural manner. We say that two V-bundles B_1 and B_2 having a given differentiable structure are equivalent if, for each l. u. s. $\{U, G, v\}$, there exists a map θ_U of U into the group of the bundle such that

(1) θ_U belongs to the same class of differentiability as the bundles B_1 and B_2.

(2) If $g \in G$ and $z \in U$, then

$$\eta^1{}_U(g)(z)\theta_U(z) = \theta_U(g^{-1}z)\eta^2{}_U(g)(z)$$

and

$$\xi^1{}_\lambda(\lambda(z))\theta_{U'}(\lambda(z)) = \theta_U(z)\xi^2{}_\lambda(\lambda(z)),$$

where $\eta^1{}_U$, $\xi^1{}_\lambda$ and $\eta^2{}_U(g)$, $\xi^2{}_\lambda$ are corresponding functions for B_1 and B_2 respectively.

Now suppose that B is a V-bundle, the group of which is some linear group. We say that B is an absolute V-bundle if, in addition to the requirements stipulated in [2], the following condition is satisfied: If $\{U, G, v\}$ is a l. u. s. and $g \in G_x$, then $h_U(g)b = b$ for each b in the fibre of B_U over x. We can give a useful characterization of absolute bundles (up to equivalence). Let B be an absolute V-bundle. Let $\{U, G, v\}$ be a l. u. s. and define $\theta_U(z) = (\text{ord } G)^{-1} \sum_{g \in G} \eta_U(g)(z)$. Since B is an absolute V-bundle, $\eta_U(g)(z)$ is the identity if $gz = z$, and therefore if $y \in U$ and $G_y = G$, θ_U defines a mapping of a small neighborhood of y into the group of the bundle. By choosing a sufficiently small l. u. s. $\{U, G, v\}$, we may assume that θ_U is defined in all of U. It is easily seen that, for $g \in G$, $\theta_U(g^{-1}z)\eta_U(g)(z) = \theta_U(z)$. Therefore, if we define $\xi^*{}_\lambda(\lambda(z)) = \theta_U(z)\xi_\lambda(\lambda(z))\theta_{U'}(z))^{-1}$, it is clear that $\xi^*{}_\lambda(\lambda(gz)) = \xi^*{}_\lambda(\lambda(z))$. Hence the system of functions $\xi^*{}_\lambda$ and $\eta_U(g)^*$, defined by $\eta_U(g)^* = \text{identity}$, define a V-bundle B^* equivalent to B. Hence, up to equivalence, *an absolute V-bundle B may be naturally regarded as a bundle over $\mathfrak{V}$ itself whose coordinate transformations are given by functions of a certain differentiability class on intersections of neighborhoods for B on $\mathfrak{V}$.*

Let B be a complex line bundle over $\mathfrak{V}$ and let α be a C^∞ metric for B. Then for each l. u. s. α_U is a C^∞ positive, real-valued function, and it is clear (using m(i), section 2 of [2]) that the differential form $\phi_U = \frac{i}{2\pi} d'd'' \log \alpha_U{}^{-1}$ is G-invariant in U and that the collection of differential forms $\{\phi_U\}$ defines a closed differential form ϕ on $\mathfrak{V}$. By Section 2, ϕ defines a certain 2-dimensional cohomology class on $\mathfrak{V}$, which we call the characteristic class of B. It is not hard to see that this cohomology class is independent of the

choice of metric. Moreover, ϕ is Hermitian, and if ϕ is positive definite, $\phi > 0$, ϕ provides a Kähler metric for $\mathcal{V}$; in this case ϕ is called a Hodge metric, and B is said to be positive (or ample), $B > 0$, if a metric α can be found such that ϕ is positive. Our principal result, stated more precisely in the very last section, is that, if there exists a positive complex line bundle over the compact, complex analytic V-manifold $\mathcal{V}$, then $\mathcal{V}$ is an algebraic variety.

Equivalence classes of complex line bundles over $\mathcal{V}$ form an additive group in an evident manner (see [16]).

If $\boldsymbol{D}$ is a divisor on $\mathcal{V}$, $\boldsymbol{D}$ naturally defines a complex line bundle $\{\boldsymbol{D}\}$ over $\mathcal{V}$. Namely, if $\{\{U, G, v\}\}$ is a collection of l.u.s.'s such that $\{\tilde{U}\}$ is a base of neighborhoods for $\mathcal{V}$, and if $\boldsymbol{D}_U$ is the divisor of the function f_U in each U, we define $\eta_U(g)(z) = (f_U \circ g^{-1})(z)/f_U(z)$, $g \in G$, and if $\lambda : U \to U'$ is an injection, we define $\xi_\lambda(\lambda(z)) = f_U(z)/(f_{U'} \circ \lambda)(z)$. It is clear that $\eta_U(g)$ and ξ_λ are non-zero holomorphic functions and satisfy the necessary conditions for defining a bundle. In particular, if $\boldsymbol{D}$ is an absolute divisor, we may assume each f_U is G-invariant so that each $\eta_U(g) = 1$ in U and therefore $\{\boldsymbol{D}\}$ is an absolute bundle. The characteristic class of $\{\boldsymbol{D}\}$ is also called the characteristic class of $\boldsymbol{D}$. Another motivation for the given definition of an absolute divisor is that it may be viewed locally as a divisor over the ring of local holomorphic functions at a point on $\mathcal{V}$. Just as it was shown for any divisor $\boldsymbol{D}$ over $\mathcal{V}$ that $m_0\boldsymbol{D}$ is an absolute divisor, where m_0 is the degree $\mathcal{V}$, so it can be shown for any complex line bundle B over $\mathcal{V}$ that $m_0 B$ is absolute. In fact, if $\{U, G, v\}$ is a l.u.s. and η_U is the corresponding function defined for B, then $\eta_U(g)(z)$ is an m_0-th root of unity whenever $gz = z$, $g \in G$, $z \in U$. These facts correspond roughly to the fact that if K is an algebraic extension of the field k, then the norm of an ideal in K is an ideal in k. In fact, in many cases of interest the divisors over $\mathcal{V}$ appear as divisors invariant under the fundamental group of a kind of universal covering space of $\mathcal{V}$.

Let $f: z \to f(z)$ be a complex analytic mapping, with non-vanishing Jacobian, from an open set $\mathcal{O}$ in C^n into C^n, let the coordinates of $f(z)$ be $f^1(z), \cdots, f^n(z)$, and let the coordinates in $\mathcal{O}$ be $z^1, \cdots, z^n$. Then we write

$$J(f)(p) = [\det(\partial f^i/\partial z^j)]_p^{-1}, \quad p \in \mathcal{O}.$$

We define the *canonical bundle* K over $\mathcal{V}$ to be the complex line bundle over $\mathcal{V}$ for which the functions η_U and ξ_λ are defined as follows:

$$\eta_U(g)(z) = J(g^{-1})(z)$$

and

$$\xi_\lambda(\lambda(z)) = J(\lambda^{-1})(\lambda(z)).$$

Let γ be a Hermitian metric for $\mathfrak{V}$. Then for each l.u.s. $\{U, G, v\}$, γ is a positive definite Hermitian matrix function $(\gamma_{U\alpha\beta'})$[3] and direct calculation shows that the system of functions $a_U = (\det(\gamma_{U\alpha\beta'}))^{-1}$ defines a metric a for the bundle K; hence the collection of forms

$$\phi_U = \tfrac{i}{2\pi} d'd'' \log(\det(\gamma_{U\alpha\beta'}))$$

defines a closed form ϕ in the characteristic class of K. The form R of type $(1,1)$, defined by $\frac{i}{2\pi}R = \phi$, we call the Ricci curvature of the metric γ. For a given l.u.s. $\{U, G, v\}$

$$R_U = \Sigma R_{U\lambda\mu'} dz^\lambda d\bar{z}^\mu = d'd'' \log(\det(\gamma_{U\alpha\beta'})).$$

In the future we will have use for the tensor R^* defined in U by

$$R^*{}_U{}^{\mu'}{}_{\alpha'} = \Sigma \gamma_U{}^{\lambda\mu'} R_{U\lambda\alpha'},$$

where $(\gamma_U{}^{\lambda\mu'})$ is the matrix inverse of $(\gamma_{U\alpha\beta'})$.

Let X be a C^∞ manifold and let C be a discontinuous group of C^∞ homeomorphisms of X satisfying the following conditions:

(1) If $x, y \in X$ and $y \notin Gx$, there exist neighborhoods U of x and V of y such that $GU \cap GV$ is empty.

(2) X/G is compact.

(3) If $x \in X$, the subgroup G_x of G leaving x fixed is finite.

(4) Each point $x \in X$ has a neighborhood U such that $gU \cap U$ is empty if $g \notin G_x$.

Then is it clear that X/G is a V-manifold. If we replace C^∞ by "complex analytic," X/G is a complex analytic V-manifold. It is clear that we obtain a V-bundle B over X/G in a natural way if we let B^* be a bundle over X and prescribe an anti-isomorphism h of G into a group of bundle maps of B^* onto itself such that if $g \in G$, $x \in X$, then $h(g)$ carries the fibre over x onto the fibre over $g^{-1}x$. The modifications of these statements for the C^∞ and complex analytic cases are obvious, as is also the necessary and sufficient condition under which the bundle induced over X/G is absolute There is an obvious correspondence between the sections of B and the cross-sections of B^* invariant under $h(G)$. An important special case is obtained by letting X be a bounded domain $\mathfrak{D}$ in C^n and G a properly discontinuous group of

[3] Our tensor notation is modified in this paper from that in [14] in that here we replace $\alpha\bar{\beta}$, $\bar{\beta}$, etc., as superscripts or subscripts by $\alpha\beta'$, β', etc. respectively.

complex analytic automorphisms of $\mathcal{D}$. We let B^* be the product bundle $\mathcal{D} \times C$ and define $h(g)(z,\zeta) = (g^{-1}z, J(g^{-1})(z)\cdot\zeta)$. Then a metric for B^* invariant under $h(g)$ is given by $\mathfrak{K}^{-1}$, where $\mathfrak{K}$ is the kernel function (see [3]), and since $d'd''\log\mathfrak{K}$ is the G-invariant (positive definite) Bergman metric, we see that the complex line bundle B over $\mathcal{D}/G$ induced by B^* and h is positive. Hence it will be an immediate consequence of our main result that $\mathcal{D}/G$ is an algebraic variety. It is easily seen that B is simply the canonical bundle K for $\mathcal{D}/G$.

4. A sufficient condition for vanishing cohomology. Kodaira [14] has demonstrated by means of a method due to Bochner [5], a useful sufficient condition for the vanishing of certain cohomology groups of a compact Kähler variety with coefficients in the sheaf of germs of holomorphic p-forms with coefficients in a complex line bundle. We demonstrate here analogous results useful for our purposes. For the most part, we are able simply to copy down Kodaira's formulas.

Let $\mathcal{V}$ be a complex analytic V-manifold of complex dimension n carrying the Kähler metric γ (see [2]). Since for each l.u.s. $\{U, G, v\}$, γ defines a G-invariant Kähler metric in U, it is easily seen that we may define covariant differentiation of alternating tensors on $\mathcal{V}$ by means of the usual formulas.

Let B be a complex line bundle over X and let $\mathfrak{a}$ be a C^∞ metric for B. If $\{U, G, v\}$ is a l.u.s., we define

$$X_{\alpha\beta'} = \partial_\alpha\bar{\partial}_\beta \log \mathfrak{a}_U^{-1}, \text{ in } U,$$

where $\partial_\alpha = \partial/\partial z^\alpha$ and $\bar{\partial}_\beta = \partial/\partial\bar{z}^\beta$. Let $R^\sigma{}_{\tau\alpha'\beta} = \bar{\partial}_\alpha(\sum \gamma^{\sigma\nu'}\partial_\beta\gamma_{\tau\nu'})$ (where $\gamma^{\sigma\nu'}$ is defined by $\sum \gamma^{\sigma\nu'}\gamma_{\tau\nu'} = \delta^\sigma{}_\tau$, $\gamma_{\tau\nu'}$ being the metric tensor and $\delta^\sigma{}_\tau$ the Kronecker symbol) and let $R_{\alpha'\beta} = \sum R^\tau{}_{\tau\alpha'\beta}$ be the Ricci tensor. Define $R_{\alpha\beta'} = \bar{R}_{\alpha'\beta}$. It is easily seen that

$$R_{\alpha\beta'} = \partial_\alpha\bar{\partial}_\beta \log[\det(\gamma_{\mu\nu'})],$$

as in Section 3. In a given system of local coordinates, we denote covariant differentiation with respect to z^α and $z^{\beta'}$ by ∇_α and $\nabla_{\beta'}$ respectively. Then if ϕ is a C^∞ form of type (r,s) on $\mathcal{V}$ with coefficients in B, we define [4]

$$\xi_{U\mu'} = \mathfrak{a}_U \sum \nabla_{\mu'}\phi_{U\tau_1\cdots\tau_r\alpha'_1\cdots\alpha'_s}\phi_U{}^{\tau_1\cdots\tau_r\alpha'_1\cdots\alpha'_s},$$

[4] For reasons of naturality we have always spoken of a metric for the bundle, whereas Kodaira (see [14]) speaks of a cross-section of a certain positive real line bundle. One is simply numerically the reciprocal of the other.

as in [14]. where ϕ_U represent ϕ in U and

$$\phi_U{}^{\tau_1\cdots\tau_r\alpha'_1\cdots\alpha'_s} = \sum \gamma^{\tau_1\sigma'_1}\cdots\gamma^{\tau_r\sigma'_r}\gamma_{\beta_1\alpha'_1}\cdots\gamma^{\beta_s\alpha'_s}\bar{\phi}_{U\sigma_1\cdots\sigma_r\beta'_1\cdots\beta'_s},$$

so that ξ defined in U by $\xi_U = \sum \xi_{U\mu'}dz^{\mu'}$ is a C^∞ 1-form on $\mathcal{V}$. If, moreover, ϕ is harmonic (i.e., $\Box\phi = 0$), we have from [14] that in U

$$\begin{aligned}-\delta\xi_U &= \sum \gamma^{\lambda\mu'}\nabla_\lambda\xi_{U\mu'} \\ &= \sum s(\delta^\sigma{}_\tau[\mathrm{X}^{\beta'}{}_{\alpha'} - R^{*\beta'}{}_{\alpha'}] + rR^\sigma{}_{\tau\alpha'}{}^{\beta'})\mathfrak{a}_U\phi_{U\sigma\tau_2\cdots\tau_r\beta'\alpha'_2\cdots\alpha'_s}\bar{\phi}_U{}^{\tau\tau_2\cdots\tau_r\alpha'\alpha'_2\cdots\alpha'_s} \\ &\quad + \# = A + \#,\end{aligned}$$

where $\#$ is a non-negative quantity, and where $\mathrm{X}^{\beta'}{}_{\alpha'} = \sum \gamma^{\lambda\beta'}\mathrm{X}_{\lambda\alpha'}$ and $R^\sigma{}_{\tau\alpha'}{}^{\beta'} = \sum \gamma^{\lambda\beta'}R^\sigma{}_{\tau\alpha'\lambda}$. If we denote by 1 the function on $\mathcal{V}$ which is equal to 1 at each point, then $*1 = dV$, the $2n$-form on $\mathcal{V}$ which represents the measure associated with γ (see [2]). We now suppose $\mathcal{V}$ is compact and obtain [2]

$$\int_{\mathcal{V}} \delta\xi * 1 = \pm \int_{\mathcal{V}} d(*\xi) = 0.$$

Therefore $0 = \int_{\mathcal{V}} (A + \#)*1$, or since $\# \geqq 0$, $\int_{\mathcal{V}} A*1 \leqq 0$. We give a partial ordering to the Hermitian forms on $\mathcal{V}$ by writing $\mathrm{X}_1 > \mathrm{X}_2$ for the Hermitian forms X_1 and X_2 if $\mathrm{X}_1 - \mathrm{X}_2$ is positive definite at each point of $\mathcal{V}$. From an examination of A, we see that if the form $(\mathrm{X}_{\alpha\beta'})$ is sufficiently high in this partial ordering, then we must have $\phi \equiv 0$, for otherwise $\int_{\mathcal{V}} A*1$ would be positive. In particular, if $r = 0$, it is enough to have $\mathrm{X} - R > 0$. Hence, if there exists a metric $\mathfrak{a}$ for B such that $\mathrm{X} - R > 0$, we have, by Theorem D-K, that

$$H^s(\mathcal{V}, \Omega^0(B)) = 0, \quad s \geqq 1. \tag{1}$$

Using the notation and results of Section 3 we may assert that (1) holds if $B - K$ is a positive bundle.

We observe that if $\mathcal{V}$ carries a positive complex line bundle B, then $\mathcal{V}$ is automatically Kähler. For if $\mathfrak{a}$ is a metric for B such that the form γ, defined by $\gamma_U = id'd'' \log \mathfrak{a}_U{}^{-1}$, is positive definite, then γ exhibits a Kähler metric for $\mathcal{V}$.

Let $\mathcal{D}$ be a bounded domain in C^n and let G be a properly discontinuous group of complex analytic automorphisms of $\mathcal{D}$ such that $\mathcal{D}/G$ is compact. The canonical bundle for $\mathcal{D}/G$ is positive (see Section 3), and if k is an integer, the cross-sections of kK are just the automorphic forms of weight k,

while $\Omega^0(kK)$ is just the sheaf of germs of automorphic forms of weight k. If $k \geqq 2$, $kK - K = (k-1)K > 0$, and therefore

$$H^s(\mathcal{D}/G, \Omega^0(kK)) = 0, \quad s \geqq 1.$$

This result is well known for $n = 1$ and has been proved by Kodaira for arbitrary n in case G acts without fixed points, but to the best of our knowledge has been only conjectured in the general case (see [20], exposé XX).

5. The quadratic transform [15]. Let $\mathcal{V}$ be an n-dimensional complex analytic V-manifold and let $x_0^* \in \mathcal{V}$. Let $\{U, G, v\}$ be a l. u. s. such that $x_0^* \in \bar{U}$ and such that $v^{-1}(x_0^*)$ consists of a single point x_0. We may assume that x_0 is the origin of C^n and that G acts as a finite linear group on C^n. We denote the complex coordinates in C^n by $z^1, \cdots, z^n$ and let $U_i = \{(z^1, \cdots, z^n) \mid (z^1, \cdots, z^n) \in U, z^i \neq 0\}$. By making an appropriate choice of coordinates (we may subject $z^1, \cdots, z^n$ to an arbitrary non-singular linear transformation if necessary) we may assume that $\bigcup_{i=1}^{n} (\bigcap_{g \in G} gU_i) = U - x_0$. This is equivalent to saying that the homogeneous polnomials $\boldsymbol{P}_i(z) = \prod_{g \in G} (gz)^i$, $i = 1, \cdots, n$, $z \in C^n$. have only the trivial zero in common, where $(gz)^i$ is the i-th coordinate of gz. If x is any point of U, we let $z_x^1, \cdots, z_x^n$ denote the coordinates with center at x defined by $z_x^i(y) = z^i(y) - z^i(x)$, $y \in U$, and let G_x be the subgroup of G leaving x fixed. It is clear that the polynomials $\boldsymbol{P}_x^i(z) = \prod_{g \in G_x} (gz)_x^i$, $i = 1, \cdots, n$, $z \in C^n$, where $(gz)_x^i$ is the i-th coordinate of gz in the coordinates $\{z_x^i\}$, have only the trivial zero in common. If U_x is a sufficiently small neighborhood of x in U, such that $G_x U_x = U_x$ and $gU_x \cap U_x$ is empty if $g \in G - G_x$, we may take $\{U_x, G_x, v_x\}$ as a l. u. s. for the neighborhood $\tilde{U}_x = v(U_x)$ of $x^* = v(x)$, where v_x is the restriction of v to U_x. Then the following construction, which we carry out for x_0^* using the l. u. s. $\{U, G, v\}$ holds equally well for x^* using the l. u. s. $\{U_x, G_x, v_x\}$.

Let S be the $(n-1)$-dimensional complex projective space with homogeneous coordinates $(t^1, \cdots, t^n)$. Consider the product manifold $\Pi = U \times S$ and let W be the non-singular subvariety (of complex dimension n) of Π determined by

$$W = \{(z, t) \mid z^i t^j - z^j t^i = 0, i, j = 1, \cdots, n\}.$$

The analytic map $\pi\colon W \to U$ of W onto U given by $\pi(z, t) = z$ has the easily seen properties that it is a biregular mapping of $W - \pi^{-1}(x_0)$ onto $U - x_0$ and that $\pi^{-1}(x_0)$ is a non-singular subvariety of W which is naturally bi-

regularly equivalent to S. Therefore without material risk of confusion we denote $\pi^{-1}(x_0)$ by S. We also use S to denote the prime divisor on W determined by S.

We observe some elementary facts about W and the subvariety or divisor S of W. Let $l_i = \{(z,t) \mid (z,t) \in W, t^i = 0\}$ and let $W_i = W - l_i$. Then on W_i we may choose

$$\text{(I)} \qquad (t^1/t^i), \cdots, (t^{i-1}/t^i), z^i, (t^{i+1}/t^i), \cdots, (t^n/t^i)$$

as local coordinates;

$$\text{(II)} \qquad z^j = z^i(t^j/t^i), j \neq i, \text{ in } W_i,$$

so that

$$\text{(III)} \quad dz^1 \wedge \cdots \wedge dz^n = (z^i)^{n-1} d(t^1/t^i) \wedge \cdots \wedge dz^i \wedge \cdots \wedge d(t^n/t^i) \text{ in } W_i.$$

Let $w \in W$ and let S_w be the ideal in the ring of local holomorphic functions $\mathfrak{O}_w$ at w consisting of functions which are locally multiples of S, and for any function f holomorphic at w, let f_w denote the principal ideal (f) in $\mathfrak{O}_w$. At each point of W_i, $S_w = (z^i)_w$. Obviously $z^1, \cdots, z^n$ are holomorphic functions on W. Let P be a homogeneous polynomial in $z^1, \cdots, z_n$ of degree q. In view of (II) we may write

$$P = (z^i)^q P^*(t^j/t^i) \text{ in } W_i,$$

where P^* is a polynomial in the (t^j/t^i)'s, $j \neq i$. From this it is easy to see that if P is a non-zero, homogeneous polynomial of degree q in $z^1, \cdots, z^n$, then at any point $w \in S_i$, P is divisible by z^i to precisely the q-th power.

We now show that the group G acting on U can be lifted in a unique way to a finite group of complex analytic transformations of W; i.e., to each $g \in G$ we may assign a unique transformation g' of W such that $\pi g' = g\pi$. It will follow trivially that $g \to g'$ is an isomorphism of G onto a group G' of complex analytic transformations of W. First of all, $g \in G$ acts on the local coordinates of U as a linear transformation:

$$(gz)^i = \sum g^i{}_\alpha z^\alpha.$$

For $(z,t) \in \Pi$ we define $g'(z,t) = (gz, gt)$, where $(gt)^i = \sum g^i{}_\alpha t^\alpha$. Then $(z,t) \in W$ if and only if $g'(z,t) \in W$ since

$$(gt)^j (gz)^i - (gt)^i (gz)^j = \sum_{\alpha,\beta} g^i{}_\alpha g^j{}_\beta (t^\beta z^\alpha - t^\alpha z^\beta).$$

Thus g' gives a biregular map of W onto itself which we also denote by g'. It is obvious that $(g_1 g_2)' = g_1' g_2'$ and that the mapping $g \to g'$ is an iso-

morphism, and it is easily seen that the complex analytic transformation g' of W satisfying $\pi g' = g\pi$ is unique.

It is clear that W/G' is a V-manifold. We let p be the canonical projection of W onto W/G' and define $\theta: (W-S)/G' \to \mathcal{V} - x_0^*$ by $\theta = \phi \circ \pi \circ p^{-1}$. Then we denote by $\mathcal{Q}_{x_0^*}(\mathcal{V})$ the complex analytic V-manifold obtained from $W/G' \cup \mathcal{V} - x_0^*$ by identifying $\tilde{w} \in (W-S)/G'$ with $\theta(\tilde{w})$. When there is no danger of confusion, we denote $\mathcal{Q}_{x_0^*}(\mathcal{V})$ by $\mathcal{V}_{x_0}$. We let $\mathcal{P}$ denote the natural projection of $\mathcal{V}_{x_0}$ onto $\mathcal{V}$ (defined by $\mathcal{P}(S) = x_0^*$ and $\mathcal{P}(x) = x$ if $x \in \mathcal{V} - x_0^*$). It is clear that if B is a V-bundle on $\mathcal{V}$, then B and $\mathcal{P}$ together define a V-bundle $\mathcal{P}^{-1}(B) = B_{[x_0]}$ having the same fibre, group, and differentiable structure as B. We denote the canonical bundle on $\mathcal{V}_{x_0}$ by K_{x_0} and denote by S^* the divisor on $\mathcal{V}_{x_0}$ which is induced by the divisor S on W (this definition is legitimate since S is stable under G'). It follows from (III) that

$$K_{x_0} = (n-1)\{S^*\} + K_{[x_0]} \tag{IV}$$

As remarked in Section 3, $m_0\{S^*\}$ is an absolute bundle. It turns out that $m\{S^*\}$ is an absolute bundle, where $m = \operatorname{ord} G$. We now construct coordinate transformations for $m\{S^*\}$. Let $\mathcal{W}_i = \bigcap_{g \in G} g'W_i$, $i = 1, \cdots, n$. Since the polynomials $\boldsymbol{P}_i$, $i = 1, \cdots n$, have only the trivial zero in common, it follows that $\mathcal{W}_1, \cdots, \mathcal{W}_n$ cover W. If $a_1, \cdots, a_n$ are complex numbers, then at each $w = (z, t) \in W$ such that $\sum a_j t^j \neq 0$ we have $(\sum a_j z^j)_w = S_w$. We put $f^i(z, t) = f^i(z) = \prod_{g \in G} (gz)^i$, $i = 1, \cdots, n$. If $w = (z, t) \in \mathcal{W}_i$, then $(mS)_w = f^i{}_w$. Moreover, each f^i is G-invariant. The open sets $U_0 = \mathcal{V} - x_0$ and $U_i = \mathcal{W}_i/G'$, $i = 1, \cdots, n$, viewed as components of the identification space $\mathcal{V}_{x_0}$, cover $\mathcal{V}_{x_0}$. We define $f^0 = 1$ on all of $\mathcal{V} - x_0$. By our remarks in Section 3, the absolute bundle $\{mS^*\}$ may be viewed as the complex line bundle over $\mathcal{V}_{x_0}$ defined by the system of coordinate neighborhoods $\{U_i\}$ and coordinate transformations (or transition functions) $\{f_{ij}\}$, where f_{ij} is the non-zero holomorphic function in $U_i \cap U_j$ defined by $f_{ij} = f^i/f^j$.

If x is any point of U, we carry out the same construction for x using the l. u. s. $\{U_x, G_x, v_x\}$ and denote the entities corresponding to U_i, W, W_i, G', $\mathcal{W}_i$, l_i, K_{x_0}, $K_{[x_0]}$, f^i, f_{ij}, m, π, $\mathcal{P}$, $\mathcal{V}_{x_0}$, and S in the above construction by U_{xi}, W_x, W_{xi}, G_x', $\mathcal{W}_{xi}$, $l^x{}_i$, K_x, $K_{[x]}$, $f_x{}^i$, f_{xij}, m_x, π_x, $\mathcal{P}_x$, $\mathcal{V}_x$, and S_x respectively. Let U^* be a neighborhood of x_0 in U such that $GU^* = U^*$. Let $\mathsf{M} = \{(x, y) \mid (x, y) \in U^* \times U, y \in Gx\}$. Then by a singular G-form of class C^∞ and type (p, q) on U with respect to U^*, we mean a C^∞ form ψ on $U^* \times U - \mathsf{M}$ such that for fixed $x \in U^*$, ψ_x defined on $U - G_x$ by $\psi_x(y)$

$=\psi(x,y)$ is a G-invariant, C^∞ form of type (p,q). Using this notation we have the

THEOREM OF CONTINUITY. *There exists a neighborhood U^* of x_0 stable under G with compact closure in U, and a real, singular G-form ψ of class C^∞ and of type $(1,1)$ on U with respect to U^* such that:*

(1) *For each $x \in U^*$, ψ_x induces in a natural manner a real C^∞ form of type $(1,1)$ on $\mathcal{P}_x^{-1}(v(U)-v(x))$ which may be extended to a C^∞ form ψ_x' on $\mathcal{P}_x^{-1}(\mathcal{V}-v(x))$ by defining $\psi_x'=0$ outside $\mathcal{P}_x^{-1}(v(U)-v(x))$;*

(2) *ψ_x' has a unique extension to a C^∞ form ψ_x' on all of $\mathcal{V}_x$, and ψ_x' belongs to the characteristic class of $\{-m_x S_x^*\}$.*

Proof. We know that there exist positive real numbers $M \geqq 1$ and $\mu \leqq 1$ such that given any $x, y \in U$ and any $g \in G$ we have $\mu\,|x-y| \leqq |gx-gy| \leqq M\,|x-y|$, where $|x-y|$ is the Euclidean distance between x and y in the given coordinates on U.

Let $\epsilon > 0$ be such that U contains a closed Euclidean ball of radius 10ϵ. Let ρ be a C^∞ function on the non-negative real axis such that

$$\rho(r^2) = \begin{cases} r^2 \text{ for } r < 6\epsilon\mu, \\ \text{a non-decreasing function in } r \text{ for } 6\epsilon\mu \leqq r \leqq 7\epsilon\mu, \\ \text{a constant } C \text{ for } r > 7\epsilon\mu. \end{cases}$$

Let U^* be a neighborhood of x_0 stable under G such that U^* is contained in the Euclidean ball of radius $\epsilon\mu/M$ about x_0.

Define Λ on $U^* \times U$ by $\Lambda(x,y) = \prod_{g \in G} \rho(|x-gy|^2)$. We note that we may write $\Lambda(x,y) = \prod_\kappa [\prod_{g \in G_x} (|x-g\kappa y|^2)]$, where κ runs over a system of representatives of the right cosets of G_x in G. For fixed $x \in U^*$ we put $\psi_x(y) = \psi(x,y) = \frac{i}{2\pi} d'd'' \log \Lambda(x,y)$, $y \in U - Gx$. It is evident that $\psi_x(y)$ is a C^∞ real form of type $(1,1)$ in $U - Gx$ invariant under G, and that $\psi_x(y)$ vanishes if $|y-x_0| = |y| > 8\epsilon$. This being the case, assertion (1) is obvious.

Let $x \in U^*$ and let N be a neighborhood of x stable under G_x such that $gN \cap N$ is empty if $g \in G - G_x$ and such that $N \subset U^*$. For any real number $r > 0$, let $\beta(x,r)$ denote the open Euclidean ball of radius r about x. We choose neighborhoods N_1, N_2, N_3 of x stable under G_x and positive numbers $\delta > \delta_1 > \delta_2 > 0$ such that $\operatorname{clos}(N_3) \subset \beta(x,\delta_2)$, $\operatorname{clos}(\beta(x,\delta_2)) \subset N_2$, $\operatorname{clos}(N_2) \subset \beta(x,\delta_1)$, $\operatorname{clos}(\beta(x,\delta_1)) \subset N_1$, $\operatorname{clos}(N_1) \subset \beta(x,\delta)$, $\operatorname{clos}(\beta(x,\delta)) \subset N$. Let ρ_x be a C^∞ function on the non-negative real axis such that

$$\rho_x(r^2) = \begin{cases} r^2 \text{ for } r < \delta_2, \\ \text{a non-decreasing function in } r \text{ for } \delta_2 < r < \delta_1, \\ \text{a constant } C_x \text{ if } r > \delta_1. \end{cases}$$

We define

$$\Xi_x(y) = \tfrac{i}{2\pi} d'd'' \log[\textstyle\prod_{g \in G_x} \rho_x(|x - gy|^2)], \quad y \in N - x,$$

and define $\Xi_x(y) = 0$ if $y \in U - N$. Let $\Psi_x(y) = \sum_\kappa \kappa^* \Xi_x(\kappa y)$ in $U - Gx$, where κ again runs over a system of right coset representatives of G_x in G. Then

$$\Psi_x(y) = \tfrac{i}{2\pi} d'd'' \log \textstyle\prod_{g \in G} \rho_x(|x - gy|^2), \quad y \in U - Gx,$$

and $(\psi_x - \Psi_x)(y) = \tfrac{i}{2\pi} d'd'' \log E_x(y)$, where

$$E_x(y) = \textstyle\prod_{g \in G} [\rho(|x - gy|^2)/\rho_x(|x - gy|^2)]$$

is a non-vanishing, G-invariant, C^∞ function in U having the constant value $(C/C_x)^m$ for $|y - x_0| > 8\epsilon$, where $m = \operatorname{ord} G$. Therefore ψ_x and Ψ_x induce forms on $\mathcal{P}_x^{-1}(v(U) - v(x))$ which differ by a form X which can be extended to a C^∞ exact form (i.e., $X = d\theta$) on all of $\mathcal{V}_x$. Hence it is sufficient to show that the form Ψ_x' on $\mathcal{V}_x - \mathcal{P}_x^{-1}(v(x))$, induced on $\mathcal{P}_x^{-1}(v(U) - v(x))$ by Ψ_x and defined to be 0 outside of $\mathcal{P}_x^{-1}(v(U))$, has an extension to a form on $\mathcal{V}_x$ belonging to the characteristic class of $\{-m_x S_x^*\}$, for if the extension exists, it is obviously unique by continuity.

We define T_x on $\mathcal{V}$ by $T_x(v(y)) = \prod_{g \in G} \rho_x(|x - gy|^2)$ if $y \in U$ and $T_x(\eta) = C_x^m$ if $\eta \notin v(U)$. Then T_x is clearly a well-defined function of class C^∞ on $\mathcal{V}$ which is non-vanishing on $\mathcal{V} - v(x)$. Moreover, in N_3,

$$T_x(v(y)) = (\textstyle\prod_{g \in G_x} |x - gy|^2) C_x^{(m - m_x)}.$$

It is obvious that $\Psi_x' = \tfrac{i}{2\pi} d'd'' \log(T_x \circ \mathcal{P}_x)$ on $\mathcal{P}_x^{-1}(\mathcal{V} - v(x))$, $\circ$ denoting the composition of functions. $\mathcal{Q}_x(\mathcal{V}) = \mathcal{V}_x$ is supposed to have been constructed with the complex coordinates $z_x^1, \cdots, z_x^n$ in N_3. Hence W_x is the non-singular subvariety of $N_3 \times S$ defined by $z_x^i t^i - z_x^i t^j = 0$, $i, j = 1, \cdots, n$; and $S_x = \pi_x^{-1}(x)$ is the non-singular prime divisor on W_x such that $(S_x)_w = (\sum a_j z_x^j)_w$ for each $w = (z, t) \in W_x$ such that $\sum a_j t^j \neq 0$. For brevity let $(z_x^1, \cdots, z_x^n) = z_x$. We also write $(gz_x)^i(y) = z_x^i(gy)$ for $g \in G_x$, so that $(gz_x)^i = \sum g^i{}_j z_x^j$ since G_x acts linearly in the coordinates $\{z_x^i\}$. We now construct a C^∞ metric $\{a_i\}$ for the bundle $\{-m_x S_x^*\}$, where a_i will be a C^∞ function in the i-th coordinate neighborhood U_{xi} of $\{-m_x S_x^*\}$. We define

$$a_i(\tilde{w})^{-1} = [\textstyle\prod_{g \in G_x} (\sum_{\nu=1}^{n} |(gt)^\nu/(gt)^i|^2)] C_x^{(m - m_x)},$$

$i=1,\cdots,n$, if $\tilde{w}$ is the natural image in $U_{xi}=\mathfrak{W}_{xi}/G_x'$ of $w=(z_x,t)\in\mathfrak{W}_{xi}$, and define $\alpha_0(\eta)^{-1}=T_x(\eta)$ if $\eta\in U_{x_0}=\mathfrak{V}_x-S_x$. Using the fact that $z_x{}^jt^i-z_x{}^it^j=0$ on W_x, $i,j=1,\cdots,n$, we obtain easily that $\alpha_i/\alpha_j=|f_{xij}|^{-2}$ on $U_{xi}\cap U_{xj}$, $i,j=0,1,\cdots,n$. Consequently, the form ϕ on $\mathfrak{V}_x$ defined on U_{xi} by $\phi=\frac{i}{2\pi}d'd''\log\alpha_i^{-1}$ is a C^∞ real form of type $(1,1)$ in the characteristic class of $\{-m_xS_x{}^*\}$, and since $\phi=\Psi_x'$ on $\mathfrak{V}_x-S_x$, our theorem is proved.

We use the notations of the statement and proof of the continuity theorem in the following discussion.

Define ω to be the form on the $(n-1)$-dimensional projective space S which is given on the affine space $t^\nu\neq 0$ by

$$\omega=\tfrac{i}{2\pi}d'd''\log(1+\textstyle\sum_{j\neq\nu}|t^j/t^\nu|^2).$$

ω is well known to be positive definite. If C^n-0 is the space of n complex variables minus the origin, ω induces the form $\omega'=\frac{i}{2\pi}d'd''\log(\sum|z^i|^2)$ on C^n-0 under the natural projection of C^n-0 onto S, and ω' is positive semi-definite. It follows from the construction of ρ and $\psi_x(y)$ that ψ_x is a positive semi-definite form in the regions $|y-x_0|<5\epsilon\mu/M$, $|y-x_0|>9\epsilon$ for each $x\in\bar{U}^*$. In the compact subspace

$$\{(x,y)\,|\,x\in\bar{U}^*,\ 5\epsilon\mu/M\leqq|y-x_0|\leqq 9\epsilon\}$$

of $\bar{U}^*\times U$, $\psi_x(y)$ depends continuously on x and y. Therefore, if α is a positive definite form in the region $5\epsilon\mu/M\leqq|y-x_0|\leqq 9\epsilon$, there exists a real number $R>0$ such that if $r\geqq R$ and $x\in\bar{U}^*$, then $r\alpha+\psi_x$ is positive definite in this region. Consequently, if α is a positive definite C^∞ form on $\mathfrak{V}$, there exists an $R>0$ such that $r\alpha+\psi_x>0$ in $\mathfrak{V}-v(x)$, for $r\geqq R$, where we use ψ_x to denote the form previously defined on $U-Gx$ as well as the form which the latter induces on $\mathfrak{V}-v(x)$ when we define $\psi_x=0$ outside $v(U)$. On $\mathfrak{W}_{xi}$ we have

$$\psi_x'=\tfrac{i}{2\pi}d'd''\log[\textstyle\prod_{g\in G_x}(\sum_{\nu=1}^{n}|(gt)^\nu/(gt)^i|^2)],$$

and the latter expression, which we denote by $\omega_i(t)$, is clearly positive definite on the subset $S_{xi}=\{t\,|\,t\in S,\prod_{g\in G_x}(gt)^i\neq 0\}$ of S. On the other hand, $r\alpha>0$ on N_3. Hence $\omega_i(t)+r\alpha>0$ on $N_3\times S_{xi}$, and if we let ι_i be the inclusion map of $\mathfrak{W}_{xi}$ into $N_3\times S_{xi}$, we have $\mathfrak{P}_x{}^*(r\alpha)+\psi_x'=\iota_i{}^*(\omega_i+r\alpha)>0$ Moreover, $\mathfrak{P}_x$ is an isomorphism of V-manifolds outside $S_x/G_x=\mathfrak{P}_x{}^{-1}(v(x))$, so that $\mathfrak{P}_x{}^*r\alpha+\psi_x'>0$ on $\mathfrak{V}_x-S_x$, and therefore $\mathfrak{P}_x{}^*r\alpha+\psi_x'>0$ on all $\mathfrak{V}_x$.

If F is a complex line bundle over $\mathfrak{V}$ and a a C^∞ metric for F, then

under the map $\mathfrak{P}_x$, F and $\mathfrak{a}$ induce, respectively, a complex line bundle F' on $\mathfrak{V}_x$ and a metric $\mathfrak{a}'$ for F'. It is trivial that $d'd''\log\mathfrak{a}' = \mathfrak{P}_x^* d'd''\log\mathfrak{a}$, where $\mathfrak{P}_x^*$ is the mapping of differential forms dual to $\mathfrak{P}_x$. Suppose now that there exists a positive complex line bundle F on $\mathfrak{V}$. Then there exists a C^∞ metric $\mathfrak{a}$ for F such that $\Upsilon = \frac{i}{2\pi} d'd''\log\mathfrak{a}^{-1}$ is positive definite on $\mathfrak{V}$. There exists a positive integer $l(x_0)$ such that $l(x_0)\Upsilon + \psi_x > 0$ in the region $5\epsilon\mu/M \leqq |y - x_0| \leqq 9\epsilon$, for all $x \in U^* = U^*(x_0)$, so that $\mathfrak{P}_x^*(l(x_0)\Upsilon) + \psi_x' > 0$ on $\mathfrak{V}_x$. We observe that $\mathfrak{P}_x^*(l(x_0)\Upsilon) + \psi_x'$ is in the characteristic class of $l(x_0)F_{[x]} - \{m_x S_x^*\}$.

We now assume that $\mathfrak{V}$ is compact and let k be any integer > 0. Let $F_1 = l(x_0)F$ and put $F^* = (n + q(x_0))F_1$, where $q(x_0)$ is an integer at our disposal. Then by (IV),

$$m_0(F^*_{[x]} - k\{m_x S_x^*\}) - m_0 K_x = [km_0 + (n-1)p_x](F_{1[x]} - \{m_x S_x^*\}) + [m_0(n + q(x_0)) - km_0 - (n-1)p_x]F_{1[x]} - m_0 K_{[x]},$$

where $p_x m_x = m = \operatorname{ord} G$. p_x is a divisor of m, so the number of distinct p_x's for $x \in U$ is finite. Choose $q(x_0)$ such that

$$[m_0(n + q(x_0)) - km_0 - (n-1)p_x]F_1 - m_0 K > 0$$

for all $x \in U$. Then $m_0(F^*_{[x]} - k\{m_x S_x^*\}) - m_0 K_x > 0$, since $F_{1[x]} - \{m_x S_x^*\} > 0$, and therefore $(F^*_{[x]} - k\{m_x S_x^*\}) - K_x > 0$ for all $x \in U^*$; for if $\mathfrak{a}$ is a metric for $m_0(F^*_{[x]} - k\{m_x S_x^*\}) - m_0 K_x$, $\mathfrak{a}^{1/m_0}$ is a metric for $(F^*_{[x]} - k\{m_x S_x^*\}) - K_x$. Thus we obtain

AUXILIARY THEOREM I. *Let $\mathfrak{V}$ be a compact, complex analytic V-manifold. Assume there exists a positive line bundle on $\mathfrak{V}$. Choose a finite number of points $x_0^1, \cdots, x_0^p$ with neighborhoods $U^*(x_0^i)$ (as above) such that $\mathfrak{V}$ is covered by the natural images of $U^*(x_0^i)$, $i = 1, \cdots, p$, and for each i let an integer $k_i \geqq 0$ be given. Then there exists a positive complex line bundle F on $\mathfrak{V}$ such that*

$$(F_{[x]} - k_i\{m_x S_x^*\}) - K_x > 0$$

for $x \in U^*(x_0^i)$, $i = 1, \cdots, p$.

Let $x^*, y^* \in \mathfrak{V}$, $x^* \neq y^*$. We construct $\mathfrak{Q}_{x^*}(\mathfrak{V})$, denote $\mathfrak{P}_x^{-1}(y^*)$ again by y^* and construct $\mathfrak{Q}_{y^*}(\mathfrak{Q}_{x^*}(\mathfrak{V}))$. We let S_x^* and S_y^* denote, respectively, the divisors on $\mathfrak{Q}_{y^*}(\mathfrak{Q}_{x^*}(\mathfrak{V}))$ associated (as before) with the $(n-1)$-dimensional complex projective spaces by which x and y are replaced in local uniformizing systems $\{U_1, G_1, v_1\}$ and $\{U_2, G_2, v_2\}$ such that $v_1(x) = x^*$ and

$v_2(y) = y^*$. These definitions are legitimated by the fact that if $\mathfrak{V}$ is a V-manifold, and $\mathfrak{V}_x$, S_x, and $\mathfrak{P}_x$ have their previous meanings, then $\mathfrak{P}_x$ is an isomorphism of $\mathfrak{V}_x - S_x/G_x'$ onto $\mathfrak{V} - x^*$. We denote by K_{xy} the canonical bundle on $\mathfrak{Q}_{y^*}(\mathfrak{Q}_{x^*}(\mathfrak{V}))$. Then computations entirely similar to those above yield (see Kodaira [15], p. 33, Lemma 2):

AUXILIARY THEOREM II. *Let $\mathfrak{V}$ be a compact, complex analytic V-manifold and assume there exists a positive complex line bundle F on $\mathfrak{V}$. Then for all sufficiently high multiples mF and all $x^*, y^* \in \mathfrak{V}$, such that $x^* \neq y^*$, we have*

$$(mF_{[xy]} - \{m_x S_x^*\} - \{m_y S_y^*\}) - K_{xy} > 0,$$

where $F_{[xy]}$ is the bundle induced on $\mathfrak{Q}_{y^*}(\mathfrak{Q}_{x^*}(\mathfrak{V}))$ by F under the mapping $\mathfrak{P}_x\mathfrak{P}_y$, and m_x and m_y have their usual meanings.

6. Holomorphic cross-sections of a positive complex line bundle having prescribed properties. The results of this section are established for absolute complex line bundles, the question of their validity for general complex line bundles having no immediate importance for our purposes. Let $\mathfrak{V}$ be a compact V-manifold. We shall suppose that there exists a positive complex line bundle over $\mathfrak{V}$, from which it follows immediately that there exists a positive absolute complex line bundle over $\mathfrak{V}$. In what follows we will always assume that a given absolute complex line bundle is defined by a system of neighborhoods $\{\tilde{U}_i\}$, where $\{U_i, G_i, v_i\}$ is a l. u. s., and a system of holomorphic coordinate functions $\{f_{ij}\}$; then for each $g \in G_i$, the map $h_{U_i}(g)$ of $U_i \times C$ onto itself is given by $h_{U_i}(g)(z, c) = (g^{-1}z, c)$ (see [2]). Thus a holomorphic cross-section of the complex line bundle is given by prescribing, for each i, a G_i-invariant holomorphic function f_i in U_i such that on $\tilde{U}_i \cap \tilde{U}_j$, $f_i = f_{ij}f_j$. If we are given any finite number of absolute complex line bundles, we may assume they have a common system of such coordinate neighborhoods.

BASIC EXISTENCE THEOREM. *Using the above notation and the notation of Section 5, let $x \in U_l$ and let k be a positive integer. Suppose that F is a positive complex line bundle on $\mathfrak{V}$ such that $(F_{[x]} - k\{m_x S_x\}) - K_x > 0$. Let $P(z_x^1, \cdots, z_x^n)$ be a polynomial invariant under G_x of total degree $< km_x$ in the coordinates $\{z_x^i\}$ with center at x. Then there exists a holomorphic cross-section $f = \{f_i\}$ of F such that $P - f_l$ has a zero of order $\geqq km_x$ at x.*

Proof. First of all, it is easily seen that it suffices to prove our theorem

for a bundle complex analytically V-equivalent to F. In what follows, we denote all members of a family of complex analytically V-equivalent bundles by the same letter. Let $\{V_e'\}$ be a common system of coordinate neighborhoods for the three absolute complex line bundles $\{m_x S_x^*\}$, $F_{[x]}$ and $F_{[x]} - \{m_x S_x^*\}$ on $\mathcal{V}_x$. Let $U_l' = \mathcal{P}_x^{-1}(\tilde{U}_l)$. Then $S_x/G_x' \subset U_l'$ and it is clear that $F_{[x]}$ is a product bundle over U_l'. Hence a holomorphic section σ of $F_{[x]}$ over any open subset $\mathcal{O}$ of U_l' may be viewed as a single holomorphic function σ_l on $\mathcal{O}$.

If Ω is any sheaf over $\mathcal{V}$ or $\mathcal{V}_x$, then in what follows we denote the stalk of Ω over the point y^* by Ω_{y^*}. We have defined the two sheaves $\Omega^0(F_{[x]})$ and $\Omega^0(F_{[x]} - k\{m_x S_x^*\})$ which are the sheaves of germs of holomorphic cross-sections of the two bundles. We now define an isomorphism of $\Omega^0(F_{[x]} - \{m_x S_x^*\})$ onto $\Omega^0(F_{[x]})$. This isomorphism arises by multiplication by a suitable cross-section of the bundle $k\{m_x S_x^*\}$. Let $y^* \in \mathcal{V}_x$ and let $\alpha_{y^*} \in \Omega^0(F_{[x]} - \{m_x S_x^*\})_{y^*}$. Let $\{U', G', v'\}$ be a l. u. s. on $\mathcal{V}_x$ such that $y^* = v'(y)$. Then in a sufficiently small neighborhood V of y stable under G'_y, such that $gV \cap V$ is empty if $g \in G' - G'_y$, a representative of α_{y^*} is a collection of holomorphic functions α^e, α^e being defined in $v'^{-1}(V_e') \cap V$ and invariant under G'_y, such that $\alpha^e = (b_{ef} \circ v')\alpha^f$ in $V \cap v'^{-1}(V_e') \cap v'^{-1}(V_f')$, where b_{ef} is the non-vanishing holomorphic coordinate function for $F_{[x]} - \{m_x S_x^*\}$ in $V_e' \cap V_f'$. Let s^e be the chosen invariant holomorphic function in V such that $(s^e)_w = (k m_x S_x^*)_w$ at each $w \in v'^{-1}(V_e')$. Define $\beta^e = s^e \alpha^e$. Then since $\beta^e = (s^e/s^f)(b_{ef} \circ v')\beta^f$ in $v'^{-1}(V_e') \cap v'^{-1}(V_f') \cap V$, $\{\beta^e\}$ represents a germ $\beta_{y^*} \in \Omega^0(F_{[x]})_{y^*}$. We define $\mathfrak{J} : \Omega^0(F_{[x]} - k\{m_x S_x^*\}) \to \Omega^0(F_{[x]})$ by $\mathfrak{J}(\alpha_{y^*}) = \beta_{y^*}$, as given above. It is clear that $\mathfrak{J}$ is an isomorphism of the sheaf $\Omega^0(F_{[x]} - k\{m_x S_x^*\})$ into the sheaf $\Omega^0(F_{[x]})$. Thus we have the exact sequence of sheaves

$$0 \to \Omega^0(F_{[x]} - k\{m_x S_x^*\}) \xrightarrow{\mathfrak{J}} \Omega^0(F_{[x]}) \xrightarrow{j} \Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})] \to 0$$

and from this we obtain the exact cohomology sequence

$$0 \to \Gamma(F_{[x]} - k\{m_x S_x^*\}) \xrightarrow{\mathfrak{J}^*} \Gamma(F_{[x]}) \xrightarrow{j^*} \Gamma(\Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})]) \xrightarrow{\delta^*} H^1(\mathcal{V}_x, \Omega^0(F_{[x]} - k\{m_x S_x^*\})) \to \cdots$$

where Γ in each case stands for "(holomorphic) sections of." However, by hy-

pothesis, $(F_{[x]} - k\{m_x S_x^*\}) - K_x > 0$. Hence $H^1(\mathfrak{V}_x, \Omega^0(F_{[x]} - k\{m_x S_x^*\})) = 0$. Therefore j^* is "onto."

We now examine the image of $\mathfrak{J}$ in $\Omega^0(F_{[x]})$. First of all, it is evident that if $\beta_{y^*} \in \mathrm{Im}(\mathfrak{J})_{y^*}$, $\beta_{y^*} = \{\beta^e\}$, then each β^e is divisible by s^e at y and so belongs to the ideal $(km_x S_x)_y$ at y. On the other hand. suppose $\beta_{y^*} = \{\beta^e\}$, where each β^e is divisible by s^e in some small neighborhood V of y stable under G'_y; we define $\alpha^e = \beta^e/s^e$ in $V \cap v'^{-1}(V_e')$. It is clear that the system $\{\alpha^e\}$ defines an $\alpha_{y^*} \in \Omega^0(F_{[x]} - k\{m_x S_x^*\})_{y^*}$ and that $\beta_{y^*} = \mathfrak{J}(\alpha_{y^*})$. Hence $\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})]$ is the subsheaf of $\Omega^0(F_{[x]})$ consists of those germs β such that a representative of β is divisible by s^e at $p(\beta)$, where p is the projection of $\Omega^0(F_{[x]})$ onto $\mathfrak{V}_x$.

We wish to study the module

$$\Gamma(\Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})]).$$

In the first place, in a coordinate neighborhood V_e' of the bundle $\{m_x S_x^*\}$, the holomorphic function s^e is a unit at $y^* \in V_e'$ if $y^* \notin \mathfrak{P}_x^{-1}(x^*)$. Therefore $[\Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})]]_{y^*} = 0$ if $y^* \notin \mathfrak{P}_x^{-1}(x^*)$.

We return to our notation of Section 5, where $\{U_x, G_x, v_x\}$, π_x, W_x, etc. were defined, and exhibit a natural decomposition of the local ring of holomorphic functions on W_x at each point of S_x. Let

$$(z_x, t)_0 = (0, \cdots, 0, t_0^1, \cdots, t_0^n) \in \mathfrak{W}_{xi} \cap S_x.$$

Let $t_0^j/t_0^i = a_0{}^j$. Then $z_x{}^i$ and $[(t^j/t^i) - a_0{}^j]$, $j \neq i$, may be taken as local coordinates with center $(z_x, t)_0$. Let f be holomorphic on W_x at $(z_x, t)_0$. Then f is a power series in these coordinates:

$$\begin{aligned} f(z_x, t) &= \textstyle\sum_{m_i; m_j, j\neq i} a_{m_i; m_1 \cdots m_i' \cdots m_n} (z_x{}^i)^{m_i} \prod_{j\neq i} [(t^j/t^i) - a^j{}_0]^{m_j} \\ &\quad \text{(where } m_i' \text{ means "omit } m_i\text{")} \\ &= \textstyle\sum_{m_i < km_x; m_j, j\neq i} a_{m_i; m_1 \cdots m_i' \cdots m_n} (z_x{}^i)^{m_i} \prod_{j\neq i} [(t^j/t^i) - a^j{}_0]^{m_j} \\ &\quad + \textstyle\sum_{m_i \geqq km_x; m_j, j\neq i m_i; m_1 \cdots m_i' \cdots m_n} (z_x{}^i)^{m_i} \prod_{j\neq i} [(t^j/t^i) - a^j{}_0]^{m_j} \\ &= \mathsf{A}(f) + \mathsf{B}(f). \end{aligned}$$

This decomposition is, of course, unique in the given local coordinates at $(z_x, t)_0$ (by the uniqueness of the power series expansion of an analytic function). Moreover, suppose N is a small neighborhood of $(z_x, t)_0$ in which this series converges, $N \subset \mathfrak{W}_{xi}$, and let $(z_x, t)_1 = (0, \cdots, 0, t_1^1, \cdots, t_1^n) \in N \cap S_x$. Then the coefficient of $(z_x{}^i)^{m_i}$ in the power series development of f at $(z_x, t)_1$ is a power series in the coordinates $[(t^j/t^i) - (t_1{}^j/t_1{}^i)]$, and so the decomposition of f into the sum of two analytic functions is the same

as that above, in a neighborhood of $(z_x, t)_1$; this expresses, so to speak, the fact that the decomposition is "analytically continuous" on S_x. $(gt)^i \neq 0$ if $(z_x, t) \in \mathfrak{W}_{xi}$ for all $g \in G_x$, and $((gz)_x{}^i)_w = (S_x)_w$ at each $w \in \mathfrak{W}_{xi}$. It is thus easily seen that the decomposition is independent of the choice of i such that $(z_x, t)_0 \in \mathfrak{W}_{xi}$, and that the decomposition is invariant under G_x'. It is evident that $\mathsf{B}(f)_{w_0} \subset (km_xS_x)_{w_0}$ at $w_0 \in S_x$, while $\mathsf{A}(f)_{w_0} \subset (km_xS_x)_{w_0}$ only if $\mathsf{A}(f) = 0$. As we have pointed out, $F_{[x]} | U_i'$ is complex analytically the product bundle, so that, if $w_0{}^* \in U_i'$, we may regard $\Omega^0(F_{[x]})_{w_0{}^*}$ as being the ring $\mathfrak{O}_{w_0{}^*}$ of local holomorphic functions at $w_0{}^*$, and if $w_0{}^*$ is the natural image in $\mathfrak{V}_x$ of $w_0 \in W_x$, $\mathfrak{O}_{w_0{}^*}$ is to be identified with the ring of local holomorphic functions at w_0 invariant under G'_{w_0}, the subgroup of G_x' leaving w_0 fixed. For any function f holomorphic in a neighborhood of y^*, denote by $f_{[y^*]}$ the germ of holomorphic functions induced by f at y^*. Then since the decomposition $f \to \mathsf{A}(f) + \mathsf{B}(f)$ is invariant under the elements of G'_{w_0}, the mapping $f \to \mathsf{A}(f)$ induces a natural homomorphism $\mathsf{A}_{w_0{}^*}$ of $\Omega^0(F_{[x]})_{w_0{}^*}$ onto a submodule $\mathfrak{a}_{w_0{}^*}$ of the complex module of local holomorphic functions at $w_0{}^*$ for $w_0 \in S_x$; and we then have the isomorphism

$$i_{w_0{}^*}\colon [\Omega^0(F_{[x]})/\mathfrak{d}[\Omega^0(F_{[x]} - k\{m_xS_x{}^*\})]]_{w_0{}^*} \cong \mathfrak{a}_{w_0{}^*}$$

defined by choosing a representative f of a coset α of

$$\mathfrak{d}[\Omega^0(F_{[x]} - k\{m_xS_x{}^*\})]_{w_0{}^*}$$

in $\Omega^0(F_{[x]})_{w_0{}^*}$ and defining $i_{w_0{}^*}(\alpha)$ to be $\mathsf{A}(f)_{[w_0{}^*]}$, under the identifications agreed upon above. We define the sheaf $\mathfrak{a}$ on $\mathfrak{V}_x$ to be the sheaf whose stalk over $w^* \in \mathfrak{P}_x{}^{-1}(x^*)$ is $\mathfrak{a}_{w^*}$, whose stalk over $y^* \notin \mathfrak{P}_x{}^{-1}(x^*)$ is 0, and a basis for whose topology is obtained by admitting the following sections:

(1) If $w_0{}^* \in \mathfrak{P}_x{}^{-1}(x^*)$ and f is holomorphic in the neighborhood N of $w_0{}^*$, we admit the section σ defined by $\sigma(w^*) = \mathsf{A}(f)_{[w^*]}$ if $w^* \in N \cap \mathfrak{P}_x{}^{-1}(x^*)$, $\sigma(w^*) = 0$ otherwise.

(2) If $y^* \in \mathfrak{V}_x - \mathfrak{P}_x{}^{-1}(x^*)$ and N is a neighborhood of y^* such that $N \cap \mathfrak{P}_x{}^{-1}(x^*)$ is empty, the only possible section of $\mathfrak{a}$ over N is the zero section.

If $y^* \notin \mathfrak{P}_x{}^{-1}(x^*)$, we define A_{y^*} and i_{y^*} by $\mathsf{A}_{y^*}\colon \Omega^0(F_{[x]})_{y^*} \to \mathfrak{a}_{y^*} = 0$ and $i_{y^*}\colon [\Omega^0(F_{[x]})/\mathfrak{d}[\Omega^0(F_{[x]} - k\{m_xS_x{}^*\})]] \to \mathfrak{a}_{y^*} = 0$. Thus we obtain the sheaf homomorphism $\mathsf{A}\colon \Omega^0(F_{[x]}) \to \mathfrak{a}$ defined by $\mathsf{A}(\alpha) = \mathsf{A}_{y^*}(\alpha)$ if $\alpha \in \Omega^0(F_{[x]})_{y^*}$, while, since the decomposition $f \to \mathsf{A}(f) + \mathsf{B}(f)$ is "analytically continuous" on S_x, i defined by $i(\alpha) = i_{y^*}(\alpha)$ if

$$\alpha \in [\Omega^0(F_{[x]})/\mathfrak{d}[\Omega^0(F_{[x]} - k\{m_xS_x{}^*\})]]_{y^*}$$

is a sheaf isomorphism, $i\colon \Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})] \cong \mathcal{A}$. Therefore we have an isomorphism, $i^*\colon \Gamma(\Omega^0(F_{[x]})/\mathfrak{J}[\Omega^0(F_{[x]} - k\{m_x S_x^*\})]) \cong \Gamma(\mathcal{A})$. It is clear that $ij = \mathsf{A}$. Thus we have the "onto" homomorphism, $i^*j^*\colon \Gamma(F_{[x]}) \to \Gamma(\mathcal{A})$.

We now examine $\Gamma(\mathcal{A})$. Let $\sigma \in \Gamma(\mathcal{A})$. Then under our agreed upon identifications, for each $w \in S_x$, $\sigma(w)$ is a germ in the local ring of holomorphic functions at w invariant under G'_w (actually, if $w^1, \cdots, w^k$ is the complete inverse image of w^* in W_x, $\sigma(w^*)$ prescribes a local holomorphic function f^i at w^i for each i such that if $g \in G_x'$ and $gw^i = w^j$, then $(f^j \circ g)_{[w^i]} = f^i{}_{[w^i]}$). It follows easily from the continuity of σ that there exists a neighborhood M of S_x stable under G_x', and an analytic function f invariant under G_x' on M such that $\mathsf{A}(f)_{[w^*]} = \sigma(w^*)$ if $w \in S_x$. Since f is holomorphic in a neighborhood of S_x in W_x, and since S_x is a compact, complex analytic submanifold of W_x, f is constant on S_x. Therefore if we define $f^*(u) = f(\pi_x^{-1}(u))$ for $u \in \pi_x(M) \subset U_x$, we see by the theorem on removable singularities that f^* is analytic on $\pi_x(M)$ and invariant under G_x. We now examine the power series expansion of f^* at x:

$$f^*(z_x) = \sum a_{m_1 \cdots m_n} (z_x{}^1)^{m_1} \cdots (z_x{}^n)^{m_n}.$$

It follows from our construction of f and definition of A that $\mathsf{A}(f) = f$ at each point $y \in \pi_x^{-1}(x)$. Therefore f^* is a polynomial of degree $< km_x$ invariant under G_x. On the other hand, if we are given an invariant polynomial f^* at x of degree $< km_x$, it is evident that f^* defines naturally a cross-section σ of $\mathcal{A}$ such that f^* is the natural image of σ under the process described above. Hence we have established a natural isomorphism $\lambda\colon \Gamma(\mathcal{A}) \cong I_x(km_x)$, where $I_x(km_x)$ is the module of invariant polynomials at x of degree $< km_x$.

Finally, we point out that there is a natural isomorphism of the holomorphic sections of F with those of $F_{[x]}$. It is obvious, of course, that we have an isomorphism ι of $\Gamma(F)$ *into* $\Gamma(F_{[x]})$. On the other hand, from the facts that $F_{[x]}$ is complex analytically the product bundle over U_l' and that S_x is compact it follows easily, again from the theorem on removable singularities, that ι is onto, $\iota\colon \Gamma(F) \cong \Gamma(F_{[x]})$. Thus we have the homomorphism $\lambda i^* j^* \iota$ of $\Gamma(F)$ onto $I_x(km_x)$, and the kernel of this homomorphism is precisely those cross-sections which, when expressed as holomorphic functions on U_l, have a power series expansion at x containing no terms of degree $< km_x$. Moveover, the image of any element of $\Gamma(F)$ is just the terms of degree $< km_x$ in this power series expansion. (Note that since G_x is linear

in $z_x^1, \cdots, z_x^n$, it preserves the degrees of the homogeneous terms.) Thus the existence theorem is proved.

COROLLARY I. *Let $x^*, y^* \in \mathcal{V}$, $x^* \neq y^*$. Let two integers $h, k > 0$ be given. In the notation of the preceding section let F be a positive bundle on $\mathcal{V}$ such that $(F_{[x]} - h\{m_x S_x^*\} - k\{m_y S_y^*\} > 0$. Let $\{U_x, G_x, v_x\}$ and $\{U_y, G_y, v_y\}$ be* l. u. s.'s *such that $v_x(x) = x^*$, $v_y(y) = y^*$, and let $\{z_x^i\}$ and $\{z_y^i\}$ be coordinates in U_x and U_y with centers at x and y respectively. Then there exists a cross-section of F which has power series expansions at x and y with the prescribed invariant terms of degrees $< hm_x$ and $< km_y$ in $\{z_x^i\}$ and $\{z_y^i\}$ respectively.*

Letting $h = k = 1$ in Corollary I, we obtain

COROLLARY II. *If $(F_{[x]} - \{m_x S_x^*\} - \{m_y S_y^*\}) - K_{xy} > 0$, there exists a pair of cross-sections σ_1 and σ_2 of F, σ_1 non-vanishing at x and y, such that $\sigma_2(x)/\sigma_1(x) = 1$ and $\sigma_2(y)/\sigma_1(y) = 0$.*

7. Imbedding theorems—The main result. Let $\mathcal{V}$ be a compact, complex analytic V-manifold, and suppose there exist a positive complex line bundle over $\mathcal{V}$. The purpose of this section is to employ the global methods and results of the preceding section together with some local results of J. P. Serre and H. Cartan [8, 19, 20] in showing how $\mathcal{V}$ may be naturally regarded as an algebraic variety.

We state some well know definitions. Let $\mathfrak{o}$ be an integral domain contained in the integral domain k and denote by $\mathfrak{o}'$ the integral closure of $\mathfrak{o}$ in k. If $\mathfrak{o} = \mathfrak{o}'$, $\mathfrak{o}$ is said to be integrally closed in k. Let V be a local analytic variety (or, more precisely, a germ of analytic varieties) at the origin of C^n. Then a representative of V is defined as the set of common zeros of some finite number of holomorphic functions in a small neighborhood of the origin. Let $\mathfrak{I}(V)$ be the ideal of V in the ring $\mathfrak{O}^{(n)}$ of local holomorphic functions at the origin. Suppose that V is irreducible, or, equivalently, that $\mathfrak{I}(V)$ is prime; V is said to be (analytically) normal if $\mathfrak{O}^{(n)}/\mathfrak{I}(V)$ is integrally closed in its quotient field. Let $\mathfrak{O}^{(m)}$ be the ring of local holomorphic functions at the origin of C^m and let $f_1, \cdots, f_m$ be elements of $\mathfrak{O}^{(n)}$ which vanish at the origin; then by the ring $\mathfrak{A}(f_1, \cdots, f_m)$ *analytically generated by* $f_1, \cdots, f_m$ we mean the subring of $\mathfrak{O}^{(n)}$ consisting of those $\varphi \in \mathfrak{O}^{(n)}$ such that $\varphi = \phi(f_1, \cdots, f_m)$ in some neighborhood of the origin of C^n, where $\phi \in \mathfrak{O}^{(m)}$ (naturally, ϕ is not necessarily unique).

We now state, in special cases and without proof, two theorems of the 1951-52 Cartan seminar [19] (these are Theorems 5 and 6 of exposé XIV).

We have somewhat specialized the statements of these theorems, as found in the seminar notes, to suit our own particular needs and have labeled them as Theorems A and B.

THEOREM A. *Using the above notation, let* $f_1, \cdots, f_m$ *be functions belonging to* $\mathfrak{O}^{(n)}$ *which vanish at the origin. Moreover, let* $f: N \to C^m$ *be the analytic mapping into* C^m *of a small neighborhood* N *of the origin in* C^n *defined by* $f(z) = (f_1(z), \cdots, f_m(z))$. *Then*

(1) *The ideal* $\mathfrak{P}$ *of those* $g \in \mathfrak{O}^{(m)}$ *for which* $g \circ f$ *vanishes in some neighborhood of the origin of* C^n *is prime; and if we denote the variety of* $\mathfrak{P}$ *by* $V(\mathfrak{P})$, *then the germ determined by the point set* $f(N)$ *at the origin is contained in* $V(\mathfrak{P})$.

(2) *If every* $\zeta \in \mathfrak{O}^{(n)}$ *is integral over* $\mathfrak{A}(f_1, \cdots, f_m)$, *then (the germ of)* $f(N) = V(\mathfrak{P})$.

THEOREM B. *Let* $f_1, \cdots, f_m \in \mathfrak{O}^{(n)}$ *and suppose that these functions vanish at the origin. Let* $\mathfrak{A}'$ *be the integral closure of* $\mathfrak{A}(f_1, \cdots, f_m)$ *in* $\mathfrak{O}^{(n)}$. *In order that* $\mathfrak{A}' = \mathfrak{O}^{(n)}$, *it is necessary and sufficient that the origin be an isolated point of the set of common zeros of* $f_1, \cdots, f_m$.

Before stating more local results of Cartan and Serre, we introduce additional notation. Let H be a finite subgroup of the general complex linear group on n variables. We may regard H as a group of automorphisms of $\mathfrak{O}^{(n)}$. Let I be the subring of those elements of $\mathfrak{O}^{(n)}$ which are invariant under H. Then we may summarize the local results which we need in the following two theorems [8, 20] which we state here without proof.

THEOREM C. *There exist a finite number of homogeneous polynomials* $Q_1, \cdots, Q_k \in I$ *such that* $\mathfrak{A}(Q_1, \cdots, Q_k) = I$.

For any $f \in \mathfrak{O}^{(n)}$ let $\omega(f)$ be the degree of the monomials of lowest degree appearing in the power series expansion of f at the origin.

THEOREM D. *Let* $Q_1, \cdots, Q_k$ *be homogeneous polynomials belonging to* I *such that* $\mathfrak{A}(Q_1, \cdots, Q_k) = I$. *Let* $d = \max_i(\deg Q_i)$. *Let* $f_1, \cdots, f_k$ *be elements of* I *such that* $\omega(Q_i - f_i) > d$, $i = 1, \cdots, k$. *Then* $\mathfrak{A}(f_1, \cdots, f_k) = I$.

Of all systems $Q_1, \cdots, Q_k$ as in Theorem D there is one for which $d = \max_i(\deg Q_i)$ is least; we denote this least value of d by $d(H)$.

Let $f_1, \cdots, f_m$ be elements of I vanishing at the origin such that $\mathfrak{A}(f_1, \cdots, f_m) = I$. We adopt all the notations of Theorem A. Then it

follows from our previous considerations and from Theorem A that $f(N)$ is an analytic variety V at the origin of C^m. Let $\mathfrak{R}$ be the local ring $\mathfrak{O}^{(m)}/\mathfrak{P}$ of V. We will establish a natural isomorphism $\mathfrak{i}: I \cong \mathfrak{R}$. Let $\varphi \in I$. Then $\varphi = \phi(f_1, \cdots, f_m)$, where $\phi \in \mathfrak{O}^{(m)}$. We let $\mathfrak{i}(\varphi)$ be the coset to which ϕ belongs in $\mathfrak{O}^{(m)}/\mathfrak{P}$. $\mathfrak{i}$ is well-defined, for if $\phi(f_1, \cdots, f_m) = \phi'(f_1, \cdots, f_m)$, $\phi - \phi'$ vanishes on V, and therefore $\phi - \phi' \in \mathfrak{P}$. It is a trivial exercise to show that $\mathfrak{i}$ is in fact a homomorphism with kernel zero and that $\mathfrak{i}(I) = \mathfrak{R}$. Since I is clearly integrally closed in its quotient field, it follows that V is (analytically) normal at the origin (of C^m).

For any $\varphi \in \mathfrak{O}^{(n)}$, we let $P_H(\varphi)$ denote the polynomial of degree $d(H)$ such that $\omega(\varphi - P_H(\varphi)) > d(H)$. Suppose that $f_1, \cdots, f_m$ are elements of I vanishing at the origin such that $P_H(f_1), \cdots, P_H(f_m)$ span the complex module of polynomials of degree $\leqq d(H)$ belonging to I which vanish at the origin. Then clearly (Theorem D) $\mathfrak{A}(f_1, \cdots, f_m) = I$. Hence in the above notation we have

LOCAL IMBEDDING THEOREM. *Let H be a finite subgroup of the general complex linear group in n variables. Let $\mathfrak{O}^{(n)}$ be the ring of holomorphic functions at the origin of C^n, and let I be the maximal subring of $\mathfrak{O}^{(n)}$ elementwise invariant under H. If $f_1, \cdots, f_m$ are elements of I vanishing at the origin such that $P_H(f_1), \cdots, P_H(f_m)$ and 1 span the complex module of polynomials in I of degree $\leqq d(H)$, then the mapping f defined by $f(z) = (f_1(z), \cdots, f_m(z))$ maps a small neighborhood of the origin of C^n onto an analytic variety V which is irreducible and (analytically) normal at the origin of C^m, and, moreover, $I \cong \mathfrak{O}^{(m)}/\mathfrak{P}(V)$.*

Let $\mathcal{V}$ be an n-dimensional V-manifold and let Φ be a complex analytic mapping of $\mathcal{V}$ into an m-dimensional complex analytic manifold M. Let $x^* \in \mathcal{V}$, let $\{U, G, v\}$ be a l.u.s. such that $v(x) = x^*$ and such that the coordinates $z^1, \cdots, z^n$ on U have center at x, and let $\zeta^1, \cdots, \zeta^m$ be local coordinates with center at $y = \Phi(x^*)$ on M. Denote by Φ' the map thus induced by Φ of a small neighborhood N of x into a small neighborhood of the origin in C^m: $\Phi'(z) = \Phi'(z^1, \cdots, z^n) = (\zeta^1, \cdots, \zeta^m)$, where $\zeta^i = f_i(z)$, $i = 1, \cdots, m$, $f_1, \cdots, f_m$ being invariant under G_x. Then we say that Φ is biregular at x^* if $f_1, \cdots, f_m$ analytically generate the whole local ring I of holomorphic functions invariant under G_x. It follows immediately from the local imbedding theorem that if Φ is biregular at x^* and if U is a small neighborhood of x^*, then $\Phi(U)$ is normal at y, the local ring of $\Phi(U)$ at y is naturally isomorphic to I, and if G_x consists of the identity alone, then Φ is biregular at x in the usual sense (note that in this case a small neighborhood

of x is an ordinary complex analytic manifold). Since I is isomorphic to the ring of local holomorphic functions on $\mathcal{V}$ at x^*, the latter is isomorphic to the local ring of $\Phi(U)$ at $\Phi(x^*)$. A sufficient condition for Φ to be biregular at x^* is that $P_{G_x}(f_1), \cdots, P_{G_x}(f_m)$ and 1 span the complex module of polynomials invariant under G_x of degrees $\leqq d(G_x)$.

We now state and prove the theorem which has been the main object of this paper.

GLOBAL IMBEDDING THEOREM. *Let $\mathcal{V}$ be a compact, n-dimensional, complex analytic V-manifold. Suppose there exists a positive complex line bundle over $\mathcal{V}$. Then $\mathcal{V}$ is an algebraic variety. More precisely, there exists a homeomorphism Φ of $\mathcal{V}$ into a complex projective space $\mathfrak{S}^m$ such that Φ is biregular at every point of $\mathcal{V}$ and $\Phi(\mathcal{V})$ is an algebraic variety in $\mathfrak{S}^m$. The mapping Φ is obtained in terms of a basis of holomorphic cross-sections of a sufficiently positive absolute complex line bundle in a manner to be described in the proof.*

Proof. Let $x_0{}^1, \cdots, x_0{}^p$ and $U^*(x_0{}^i)$, $i = 1, \cdots, p$, be chosen as in Auxiliary Theorem I of Section 5 and choose positive integers k_i, $i = 1, \cdots, p$, as follows. Let $\{U^i, G^i, v^i\}$ be a l. u. s. such that $U^*(x_0{}^i) \subset U^i$, let $G^i{}_1 = G^i, G^i{}_2, \cdots, G^i{}_{n_i}$ be the subgroups of G^i, and let $k_i > \max_j(d(G^i{}_j))$. Then by Auxiliary Theorems I and II of Section 5, there exists a positive, absolute, complex line bundle F on $\mathcal{V}$ such that

(I) $(F_{[x]} - k_i\{m_x S_x{}^*\}) - K_x > 0$, $i = 1, \cdots, p$, and

(II) $(F_{[x]} - \{m_x S_x{}^*\} - \{m_y S_y{}^*\}) - K_{xy} > 0$ for all $x^*, y^* \in \mathcal{V}$, $x^* \neq y^*$.

By Theorem F of [2], $\Gamma(F)$ is finite dimensional. Let $\sigma_0, \cdots, \sigma_m$ be a basis for $\Gamma(F)$. Since (II) is satisfied, it follows from Corollary II of Section 6 that for each $x^* \in \mathcal{V}$ at least one $\sigma_i(x^*) \neq 0$.

Let x^* belong to the coordinate neighborhhod V_k (on $\mathcal{V}$) of the (absolute) bundle F and let $\sigma_{0k}, \cdots, \sigma_{mk}$ be the holomorphic functions which represent $\sigma_0, \cdots, \sigma_m$ in V_k. The ratios $\sigma_{ik}(x^*)/\sigma_{jk}(x^*)$ (defined for all j such that $\sigma_{jk}(x^*) \neq 0$) are evidently independent of the choice of coordinate neighborhood V_k. Therefore, the mapping $\Phi: \mathcal{V} \to \mathfrak{S}^m$ defined by $\Phi(x^*) = (\sigma_{0k}(x^*), \cdots, \sigma_{mk}(x^*))$, if $x^* \in V_k$, is everywhere defined and continuous. By Corollary II of Section 6, Φ is one-to-one. Since $\mathcal{V}$ is compact, Φ is a homeomorphism of $\mathcal{V}$ onto $\Phi(\mathcal{V})$.

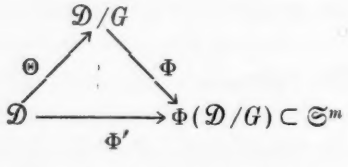
Let x^* be a point of $\mathcal{V}$ and let $\{U, G, v\}$ be a l. u. s. such that $v(x) = x^*$,

such that the coordinates $z^1, \cdots, z^n$ in U have center at x, and such that $v(U)$ is contained in a single coordinate neighborhood V_k for F. By an appropriate linear transformation on $\sigma_0, \cdots, \sigma_m$, we obtain a new basis $\sigma_0', \cdots, \sigma_m'$ for $\Gamma(F)$ such that $P_G(\sigma_0') = 1$ and $\sigma_i(x^*) = 0$ if $i > 1$; then it is clear from (I) and the Basic Existence Theorem of Section 6 that $A(\sigma_{1k}'/\sigma_{0k}', \cdots, \sigma_{mk}'/\sigma_{0k}') = I$, and therefore Φ is biregular at x^* since the above linear transformation on $\sigma_0, \cdots, \sigma_m$ amounts simply to a non-singular transformation of coordinates at $\Phi(x^*)$.

Our Local Imbedding Theorem shows that $\Phi(\mathcal{V})$ is a complex analytic subvariety of $\mathfrak{S}^m$, and, being compact, it is an algebraic variety by Chow's Theorem [9, 12, 20]. q.e.d.

It is easy to see that if Φ_1 is another projective imbedding of $\mathcal{V}$ constructed as above, then the two projective models $\Phi(\mathcal{V})$ and $\Phi_1(\mathcal{V})$ of $\mathcal{V}$ are biregularly equivalent.

We list some applications of the theorem just proved. Let $\mathcal{D}$, as in Section 3, be a bounded domain in C^n and let G be a discontinuous group of complex analytic transformations of $\mathcal{D}$ such that $\mathcal{D}/G$ is compact (see [4, 22]). Then $\mathcal{D}/G$ is a compact, complex analytic V-manifold, and, as we see from Section 3, the Bergman metric affords a positive definite form in the characteristic class of the positive complex line bundle K whose holomorphic cross-sections are to be naturally identified with the automorphic forms of weight 1. Then there is a multiple m_0K of K such that for all sufficiently large positive integers N, a basis of holomorphic cross-sections of Nm_0K (i.e., automorphic forms of weight Nm_0) affords an imbedding Φ of $\mathcal{D}/G$ into a projective space $\mathfrak{S}^m$ such that Φ satisfies the conditions of our main theorem. In other words, if $\theta_0, \cdots, \theta_m$ are a basis of the module of automorphic forms of weight Nm_0, the map $\Phi'\colon \mathcal{D} \to \mathfrak{S}^m$, defined by $\Phi'(x) = (\theta_0(x), \cdots, \theta_m(x))$, is an imbedding of $\mathcal{D}$ into $\mathfrak{S}^m$ such that $\Phi'(\mathcal{D})$ is an algebraic variety, and such that Φ' may be factored by way of $\mathcal{D}/G$:



where Φ is a homeomorphism and $\Phi' = \Phi \circ \Theta$. This result has already been obtained in the Cartan seminar of 1953-54 together with the additional information that, a priori, we only need require $\theta_0, \cdots, \theta_m$ to be a basis for the Poincaré θ-series of weight Nm_0. That proof is based on the kind of

local imbedding theorem given above and on the elementary properties of θ-series. It is nearly obvious that if we let $\{P_\alpha\}$ be the set of homogeneous polynomials in $m+1$ variables such that $P_\alpha(\theta_0, \cdots, \theta_m) \equiv 0$ in $\mathfrak{D}$, then $\Phi'(\mathfrak{D})$ is the variety in $\mathfrak{S}^m$ defined by the equations $P_\alpha(t^0, \cdots, t^m) = 0$ in the homogeneous coordinates $t^0, \cdots, t^m$.

Another example is obtained by letting X be a non-singular algebraic variety in the complex projective space $\mathfrak{S}^s$ and letting G be a finite group of complex transformations of X.[5] Then X/G is a compact, complex analytic V-manifold and a positive complex line bundle is constructed as follows:

Let S be a general non-singular hyperplane section of X. Then GS (the union of the translates of S under G) evidently gives rise to an absolute divisor S^G on X/G, and the bundle $\{S^G\}$ is positive since $\{S\}$ is a positive bundle on X, the characteristic class of $\{S\}$ being simply the class of the metric form ω induced on X by the "standard" metric of $\mathfrak{S}^s$, and a form in the characteristic class of S^G being induced by the sum of the translates of ω under G. If $X = \mathfrak{S}^s$, the imbedding is given by a basis of invariant homogeneous polynomials of a sufficiently high degree.

A third example of a V-manifold has been described by Satake, in a recent article [18], in compactifying the fundamental domain of the 2-dimensional modular group of Siegel which acts in the space of 2×2 complex symmetric matrices of positive definite imaginary part. The author of the present article has shown that there exists a positive definite bundle over this V-manifold, which is consequently an algebraic variety.

There are, of course, compact, complex analytic V-manifolds which are not algebraic—an example of such is afforded by every compact, complex analytic manifold (in the usual sense) which is not algebraic.

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ON ESTIMATING ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS IN THE L_2 NORM.*

By MARTIN SCHECHTER.[1]

1. Introduction. In recent years many authors have been treating problems in partial differential equations within the framework of Hilbert spaces involving L_2 norms. As should be expected, it is necessary to establish so called *a priori* estimates. In particular, if A is a linear m-th order partial differential operator with continuous coefficients and D^m denotes the generic m-th order derivative, it is desired to know when one has

$$\| D^m u \|^2 \leqq K(\| Au \|^2 + \| u \|^2) \tag{1}$$

for all functions u in a given class U.[2] (The functions u as well as the coefficients of A are to be complex valued and defined in the closure of a domain G. The norm is that of $L_2(G)$.)

If U is the set of infinitely differentiable functions with compact support in G, it is easily shown that ellipticity is a necessary and sufficient condition for (1) to hold.[3] On the other hand, if no boundary restrictions are placed on the u, the inequality is violated (cf. [1]).

However, as suggested by the Dirichlet problem, one can consider intermediate classes of functions, namely functions having derivatives up to a certain order vanish on the boundary. In this connection, Guseva [7], Nirenberg [6], and Browder [2] proved that (1) holds for operators of even order with real parts elliptic and for functions u having zero Dirichlet data, i.e., functions which vanish on the boundary together with all derivatives of order less than $m/2$.

It is the purpose of this paper to study the intermediate cases more fully. Let n be the dimension of G and let $C^r(G)$ be the set of functions infinitely differentiable in $\bar{G}$ having all derivatives of order less than r vanish

* Received December 17, 1956.

[1] This work was completed while the author was an Institute of Mathematical Sciences Fellow at New York University.

[2] The letter K will denote a constant independent of the functions u. It will denote a different constant in different expressions.

[3] See Lemma 9 of Section 6, this paper. Note that our definition of ellipticity (Section 2) is weaker than that of some authors (cf. [4]).

on the boundary. (If $r > 0$, it is understood that the functions themselves vanish on the boundary.) It is convenient to consider the case $n = 2$ separately. In all other cases we have the simple result: (1) holds for all $u \in C^r(G)$ if and only if A is elliptic and $m \leqq 2r$. (Precise regularity conditions assumed for the boundary of G are given in Section 6.) Moreover, if the coefficients of A are real, this result holds also for $n = 2$.

In the general case of $n = 2$ we must take into consideration the roots of the characteristic polynomial P of A. Properly defined (see Section 2) there are, at the boundary, exactly m roots of P with imaginary parts perpendicular to the boundary. Let p_1 represent the number of those with imaginary parts directed inward and p_2 the number having imaginary parts directed outward. Setting $q = \max(p_1, p_2)$ we have for $n = 2$: (1) holds for all $u \in C^r(G)$ if and only if A is elliptic and $q \leqq r$. (The reason for the simpler formulation for higher n is that in those cases ellipticity implies that m is even and that $q = m/2$. See Section 2.)

In the next section we make the necessary definitions and state our main results (Theorems 1 and 2). In proving them (Section 6) we employ a method developed by Gårding [4] and Aronszajn [1]. It consists of breaking G up into small subdomains and, after appropriate transformations, of approximating A by a homogeneous operator A' with constant coefficients. This reduces the problem to a similar one for A' in a special type of domain. The corresponding theorems for A' are stated in Section 3 and proved in Section 5 by means of Fourier transform methods introduced in Section 4.

For the special case of the Laplacian operator, inequality (1) was first proved by Ladyženskaya and Michelin for functions having zero Dirichlet data on the boundary (cf. [7]).

Aronszajn, in an unpublished work, has treated inequalities of the form (1) in which $\| Au \|^2$ is replaced by a general bilinear form. He considered functions with no boundary restrictions.

S. Agmon, A. Douglis, and L. Nirenberg have proved inequality (1) for functions satisfying very general mixed initial values and boundary conditions. Their work is as yet unpublished.

The author would like to express his sincerest gratitude to Professor Louis Nirenberg, who suggested the problem, for his invaluable assistance and expert guidance.

2. Statement of results. Let G be a bounded n dimensional domain with closure $\bar{G}$ and boundary B. We assume, for convenience, that B is

infinitely differentiable, although this is not necessary.[4] By $C^r(G)$ we shall denote the class of all infinitely differentiable complex valued functions defined in $\bar{G}$ and having all derivatives of order less than r vanish on B.

Let $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of indices, each $\geqq 0$, and set $|\mu| = \sum \mu_k$. If $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ is an n dimensional vector, we employ the notation $\xi^\mu = \xi_1^{\mu_1}\xi_2^{\mu_2}\cdots\xi_n^{\mu_n}$. Similarly, if $x = (x_1, x_2, \cdots, x_n)$ is a coordinate system in G, set

$$D_k = -i\partial/\partial x_k, \qquad D^\mu = D_1^{\mu_1}D_2^{\mu_2}\cdots D_n^{\mu_n}.$$

We shall use the following norms:

$$\| u \|^2 = \int_G | u | \, dx, \qquad (| u |_m)^2 = \sum_{|\mu|=m} \| D^\mu u \|^2, \qquad (\| u \|_m)^2 = \sum_{t \leqq m} | u |_t^2.$$

Suppose that a_μ are complex valued functions continuous in $\bar{G}$. We consider operators of the form $A = \sum a_\mu D^\mu$, where $|\mu| \leqq m$, and shall call A elliptic in $\bar{G}$ if its characteristic polynomial $P(x, \xi) = \sum a_\mu \xi^\mu$, where $|\mu| = m$, does not vanish for real vectors $\xi \neq 0$ at any point of $\bar{G}$. We are now in a position to state our first result.

THEOREM 1. *If $n > 2$, a necessary and sufficient condition that*

$$\text{(I)} \qquad \| u \|_m^2 \leqq K(\| Au \|^2 + \| u \|^2) \text{ for all } u \in C^r(G)$$

is that A is elliptic and $m \leqq 2r$.

In proving Theorem 1 we take into account the following considerations. At each point of B we have exactly m complex vector roots of P having imaginary parts perpendicular to B. They are to be found as follows: Rotate the coordinate axes until x_n is perpendicular to B at the given point. A and P are correspondingly changed. Now let $\xi_1, \xi_2, \cdots, \xi_{n-1}$ take on any real values not all zero. Then there will be m complex roots of P for ξ_n. Let p_1 be the number of those roots having positive imaginary parts and let p_2 be the number with negative imaginary parts. If A is elliptic, these quantities do not change as $\xi_1, \xi_2, \cdots, \xi_{n-1}$ are allowed to take on all real values not all zero. Moreover, continuity of the coefficients together with ellipticity will guarantee that p_1 and p_2 remain unchanged as one moves along B. Hence, for elliptic operators, the quantity $q = \max(p_1, p_2)$ is properly defined. Theorem 1 now follows from the more general

THEOREM 2. (I) *holds if and only if A is elliptic and $q \leqq r$.*

[4] Weaker conditions on B which are sufficient for the validity of the theorems stated here are discussed in Section 6.

COROLLARY 1. (I) *holds if* $m \leqq 2r$ *and there is a complex constant* γ *such that* $\mathrm{Re}\,\gamma A$ *is elliptic.*

The connection between the two theorems is seen in the

Remark. If $n > 2$, the ellipticity of A implies that m is even and that $q = m/2$.

Proof. If $n > 2$, one can continuously take the vector $(\xi_1, \xi_2, \cdots, \xi_{n-1})$ through real values into the vector $(-\xi_1, -\xi_2, \cdots, -\xi_{n-1})$ without going through the origin. By our previous argument the quantities p_1 and p_2 remain unchanged. But the signs of the corresponding roots will be reversed. Hence $p_1 = p_2$.

In the two dimensional case, q can have other values and Theorem 2 is needed. However, if the coefficients of A are real, $p_1 = p_2$ and Theorem 1 holds also for $n = 2$.

Theorems 1 and 2 will be proved in Section 6 by means of the special theorems of the next section.

3. Operators with constant coefficients. We now consider theorems similar to those of Section 2 but applying to a special type of domain and to homogeneous operators with constant coefficients.

Let G^* be an n dimensional domain having part of its boundary on the hyperplane $x_n = 0$. Denote this part by B^*_1 and the remainded by B^*_2. By $C^{*r}(G^*)$ we shall mean the class of functions u satisfying

1) u is infinitely differentiable in $\bar{G}^*$;

2) u vanishes near B^*_2;

3) all derivatives of u of order $< r$ vanish on B^*_1.

We consider an m-th order homogeneous operator with constant (complex) coefficients $A = \sum a_\mu D^\mu$, where $|\mu| = m$. If A is elliptic, its characteristic polynomial $P(\xi) = \sum a_\mu \xi^\mu$, where $|\mu| = m$, does not vanish for any real vector $\xi \neq 0$.

Corresponding to Theorem 1, we have

THEOREM 3. *For* $n > 2$,

$$|u|_m \leqq K \|Au\| \quad \textit{for all } u \in C^{*r}(G^*) \tag{II}$$

if and only if A *is elliptic and* $m \leqq 2r$.

The quantities p_1, p_2, q for A are introduced as follows: Let $\xi_1, \xi_2, \cdots, \xi_{n-1}$

take on any set of real values not all zero. There will be m complex roots of P for ξ_n. Let p_1 be the number of those with positive imaginary parts and p_2 the number with negative imaginary parts. Set $q = \max(p_1, p_2)$.

THEOREM 4. (II) *holds if and only if A is elliptic and $q \leqq r$.*

COROLLARY 2. (II) *holds if $m \leqq 2r$ and there is a complex constant γ such that $\operatorname{Re} \gamma A$ is elliptic.*

The proofs of these results will be given in Section 5.

4. Preliminaries. In this and the following section, we shall alter our notation slightly in order to facilitate the proofs. Set

$$x = (x_1, x_2, \cdots, x_{n-1}), \quad y = x_n; \qquad \xi = (\xi_1, \xi_2, \cdots, \xi_{n-1}), \quad \eta = \xi_n;$$

$$\mu = (\mu_1, \mu_2, \cdots, \mu_{n-1}); \qquad \xi x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{n-1} x_{n-1}.$$

We make use of the Fourier transform

$$u^* = (2\pi)^{-n/2} \int \exp -i(\xi x + \eta y) u(x, y)\, dx dy,$$

applied to functions $u \in C^{*r}(G^*)$. (Whenever limits are omitted, integration is to be taken over all space. We assume $u \equiv 0$ outside G^*.)

By partial integration we have

$$(D^\mu u)^* = \xi^\mu u^*, \qquad (D_n u)^* = \eta u^* + u', \tag{2}$$

where $u' = -i(2\pi)^{-n/2} \int \exp(-i\xi x) u(x, 0)\, dx$. Hence, by induction,

$$(D_n{}^k u)^* = \eta^k u^* + \sum_{s=r}^{k-1} \eta^{k-s-1} (D_n{}^s u)'. \tag{3}$$

LEMMA 1. *If $u \in C^{*1}(G^*)$, then $\int u^* d\eta = 0$.*

Proof. For $v(y)$ continuous, piecewise smooth, and in $L_1(-\infty, \infty)$,

$$v(z) = \tfrac{1}{2\pi} \int \exp(i\eta z) \int \exp(-i\eta y) v(y)\, dy d\eta.$$

Set $z = 0$ and $v(y) = \frac{1}{2\pi} \int \exp(-i\xi x) u(x, y)\, dx$.

LEMMA 2. *If $u \in C^{*r}(G^*)$, then $\int \eta^s u^* d\eta = 0$ for $0 \leqq s < r$.*

Proof. Since $(D_n{}^s u) \in C^{*1}(G^*)$, we have by Lemma 1, $\int (D_n{}^s u)^* d\eta = 0$. But $(D_n{}^s u)^* = \eta^s u^*$.

LEMMA 3. *If $H(\xi, \eta)$ is homogeneous of degree t in (ξ, η), then $F(\xi) = \int H(\xi, \eta) d\eta$ (provided it exists) is homogeneous of degree $t + 1$ in ξ.*

Proof. If $\lambda > 0$, set $\eta' = \eta/\lambda$. Then

$$F(\lambda\xi) = \int H(\lambda\xi, \eta)\, d\eta = \int H(\lambda\xi, \lambda\eta')\, d\lambda\eta' = \lambda^{t+1} \int H(\xi, \eta')\, d\eta' = \lambda^{t+1} F(\xi).$$

Consider a homogeneous m-th order operator with constant coefficients $A = \sum a_\mu D^\mu D_n^k$ and its characteristic polynomial $P(\xi, \eta) = \sum a_\mu \xi^\mu \eta^k$, where $|\mu| + k = m$. If $u \in C^{*r}(G^*)$, we have, by (2) and (3), $(Au)^* = Pu^* + S$, where $S = \sum_{t=1}^{m-r} g_t \eta^{t-1}$ and, if $|\xi|^2 = \sum_{j=1}^{n-1} |\xi_j|^2$ and $\xi \neq 0$,

$$g_t(\xi) = \sum_{k=t+r}^{m} w_{k-t} |\xi|^{k-t-1} \sum_{|\mu|+k=m} a_\mu \xi^\mu, \qquad 1 \leqq t \leqq m - r$$

$$w_s(\xi) = |\xi|^{-s-1} (D_n^s u)' \text{ if } \xi \neq 0, \; w_s(\xi) = 0 \text{ if } \xi = 0 \qquad r \leqq s < m.$$

LEMMA 4. *If $a_{00\cdots 0} \neq 0$, then $g_t(\xi_0) = 0$ for $1 \leqq t \leqq m - r$ if and only if $w_s(\xi_0) = 0$ for $r \leqq s < m$.*

Proof. Start with $t = m - r$.

LEMMA 5 (Aronszajn). *Let A be an elliptic operator and assume that for $\xi \neq 0$ and $u \in C^{*r}(G^*)$,*

$$\int |(Au)^*|^2\, d\eta = \int |Pu^* + M|^2\, d\eta + \sum H_{st} w_s \bar{w}_t,$$

where

a) *$M = \sum V_s w_s$ and each V_s is homogeneous of degree m in (ξ, η);*

b) *Each H_{st} is homogeneous in ξ of degree $2m + 1$;*

c) *For each real $\xi \neq 0$, the matrix (H_{st}) is positive definite.*

Conclusion. (II) *holds.*

Proof. Let $D^m u$ be any m-th order derivative of u. Then by (2) and (3), $(D^m u)^* = Qu^* + T$, where Q is an m-th power of (ξ, η) and T is analogous to S. Setting $J = |P|^2 - \epsilon |Q|^2$, we have

$$\begin{aligned}\int (|(Au)^*|^2 - \epsilon |(D^m u)^*|^2)\, d\eta &= \sum H_{st} w_s \bar{w}_t \\ &\quad + \int (J |u^*|^2 + 2\mathrm{Re}\, u^* (P\bar{M} - \epsilon Q\bar{T}) + |M|^2 - \epsilon |T|^2)\, d\eta \\ &= \sum H_{st} w_s \bar{w}_t + \int J |u^* + J^{-1}(\bar{P}M - \epsilon \bar{Q}T)|^2\, d\eta \\ &\quad + \int J^{-1}(J(|M|^2 - \epsilon |T|^2) - |\bar{P}M - \epsilon \bar{Q}T|^2)\, d\eta.\end{aligned}$$

Since A is elliptic, we can take ϵ so small that $J \geqq 0$. Moreover, T has the same homogeneity properties as M. Hence

$$\int (|(Au)^*|^2 - \epsilon |(D^m u)^*|^2)\, d\eta \geqq \sum H_{st} w_s \bar{w}_t - \epsilon \sum L_{st} w_s \bar{w}_t,$$

where the L_{st}, by Lemma 3, are homogeneous in ξ of degree $2m+1$. Now consider

$$\phi(\xi, w) = (\sum L_{st} w_s \bar{w}_t)/(\sum H_{st} w_s \bar{w}_t),$$

where $w = (w_r, w_{r+1}, \cdots, w_{m-1})$. Since $\phi(\lambda\xi, w) = \phi(\xi, \lambda w) = \phi(\xi, w)$ for all $\lambda > 0$, we may consider ϕ defined only on the compact set $|\xi| = |w| = 1$. Since, by hypothesis, the denominator does not vanish on this set and all functions are continuous, there is a constant K such that $\phi(\xi, w) < K$ for all ξ, w. Taking $\epsilon < K^{-1}$ we have

$$\int (|(Au)^*|^2 - \epsilon |(D^m u)^*|^2) d\eta \geqq 0 \text{ for all } \xi \neq 0.$$

Hence, by the Parseval identity,

$$\int (|Au|^2 - \epsilon |D^m u|^2) dx dy = \int (|(Au)^*|^2 - \epsilon |(D^m u)^*|^2) d\xi d\eta \geqq 0.$$

Q. e. d.

LEMMA 6. *Let $P(z)$ be a polynomial of degree $\geqq 2q$ having exactly q roots in the half plane $\operatorname{Im} z > 0$ and no real roots. If*

$$c_{st} = \int_{-\infty}^{\infty} x^{s+t}/P(x)\, dx \qquad 0 \leqq s, t < q,$$

then the $q \times q$ matrix (c_{st}) is non-singular.

Proof.[5] Set $P = P_1 P_2$, where P_1 contains only the q roots with positive imaginary parts. Suppose that

$$\sum_{t=0}^{q-1} c_{st} \lambda_t = 0 \tag{4}$$

and not all the complex numbers λ_t vanish. We can extend the integrals to be complex line integrals along a path E enclosing the q roots. Hence by (4),

$$\int_E z^s Q(z)/P(z)\, dz = 0 \qquad 0 \leqq s < q, \tag{5}$$

where $Q(z) = \sum_{t=0}^{q-1} \lambda_t z^t$. We may assume P_1 and Q relatively prime. Otherwise, the common factor can be cancelled in (5) and the resulting polynomials used. Thus there are polynomials F_1, H_1 such that $F_1 Q + H_1 P_1 = 1$, and hence there are polynomials F, H such that $FQ + HP_1 = z^{q-1} P_2$. By dividing

[5] This proof was suggested by Shmuel Agmon. It replaces the cumbersome one originally intended by the author.

F by P_1 and taking the remainder if necessary, we can demand that the degree of F be less than q. Now

$$\int_E F(z)Q(z)/P(z)dz = \int_E (FQ + HP_1)/P\, dz = \int_E z^{q-1}/P_1\, dz \neq 0.$$

But this contradicts (5). Hence all the λ_t vanish. Q. e. d.

Lemmas 1-6 are used in the next section to prove Theorem 4.

5. Proof of Theorem 4. Let G be any domain, and denote by $U(G)$ the class of infinitely differentiable functions with compact support in G. If A is any m-th order operator with constant coefficients, we have

LEMMA 7 (Gårding). *The inequality*

$$\| u \|_m^2 \leqq K(\| Au \|^2 + \| u \|^2) \text{ for all } u \in U(G) \tag{III}$$

holds if and only if A is elliptic.

Sufficiency may be proved by applying Fourier transfroms and the Parseval identity (cf. [4]). Necessity may be obtained either by a counter example [1] or a device due to Hörmander [5].

Proof of Theorem 4. *Sufficiency.* Suppose that $u \in C^{*r}(G^*)$ and let w be defined as in Section 4. Set

$$h_t(\xi) = \sum V_{st}\bar{w}_s, \qquad R = \sum_{t=1}^{m-q} h_t \eta^{t-1},$$

where each V_{st} is homogeneous in ξ of degree $2m - t + 1$. Since $m - q \leqq q \leqq r$, we have, by Lemma 2, $\int u^* R d\eta = 0$. Using the notation of Section 4, we have

$$\begin{aligned}\int |(Au)^*|^2 d\eta &= \int (|Pu^*|^2 + 2\mathrm{Re}\, u^*(P\bar{S} - cR) + |S|^2) d\eta \\ &= \int |Pu^* + \bar{P}^{-1}(\bar{P}S - c\bar{R})|^2 d\eta + \int |P|^{-2}(|PS|^2 - |\bar{P}S - c\bar{R}|^2) d\eta.\end{aligned}$$

The second integral, which equals

$$\mathrm{Re}\, c \int |P|^{-2}(2\bar{P}RS - c|R|^2) d\eta,$$

can be made positive definite in the w_s (by choosing R properly) if the same is true of

$$W = \mathrm{Re} \int SR/P\, d\eta.$$

To see this one employs the same reasoning as that used at the end of the

proof of Lemma 5. The quantity c must be taken real and sufficiently small. Thus, by Lemmas 3 and 5, we are through if we can pick R in such a way that W is positive definite in the w_s for each $\xi \neq 0$. Now the coefficient of $g_s h_t$ in W is

$$c_{st} = \int \eta^{s+t-2}/P \, d\eta,$$

and by Lemma 6 the $(m-q)\times(m-q)$ matrix (c_{st}) is non-singular. If we set $g_{m-r} = g_{m-r+1} = \cdots = g_{m-q} = 0$, we can solve the system

$$\sum_{t=1}^{m-q} c_{st} h_t = |\xi|^{2s+1} \bar{g}_s$$

for the h_t. Since the $\bar{g}_s$ are linear in the $\bar{w}_s$, so are the h_t, and by uniqueness it is easily checked that h_t is homogeneous in ξ of degree $2m-t+1$. Moreover, for this choice of R,

$$W = \sum_{s=1}^{m-r} |\xi|^{2s+1} |g_s|^2$$

and hence cannot vanish unless all the g_s do. But by the ellipticity of A, this cannot happen without all the w_s vanishing (Lemma 4). Hence W is a positive definite quadratic form in the w_s. This completes the proof.

Necessity. That A must be elliptic follows from Lemma 7. We now prove that

(IV) $$|u|_m^2 \leqq K(\|Au\|^2 + \|u\|^2) \text{ for all } u \in C^{*r}(G^*)$$

cannot hold if $q > r$. (This is slightly stronger than the statement in the theorem and will be used in the next section.) We employ an extension of an example due to Aronszajn [1]. Assume for definiteness that $q = p_1 > r$. This means that, for any $\xi \neq 0$, there are complex numbers $z_1, z_2, \cdots, z_q$ such that $\operatorname{Im} z_s > 0$ and $P(\xi, z_s) = 0$ for $1 \leqq s \leqq q$. We may assume that G^* is so situated that B^*_1 contains the origin (see Section 3). Let ϕ be a function in $C^{*0}(G^*)$ which is identically 1 near the origin and such that $\partial\phi/\partial y = 0$ on $y = 0$. We first assume that all the z_s are different. Then we can find a set of q numbers b_s, not all zero, such that

$$\sum_{s=1}^{q} b_s z_s^t = 0 \qquad \text{for } 0 \leqq t < q-1.$$

Set $v(x,y) = \sum_{s=1}^{q} b_s (\xi x + z_s y + \epsilon i)^{\frac{1}{2}}$ and $u = \phi v$. Then $u \in C^{*r}(G^*)$, since

$$\partial^t u/\partial y^t \propto \phi \sum b_s z_s^t (\xi x + z_s y + \epsilon i)^{\frac{1}{2}-t}$$

which vanish on $y=0$ for $0 \leqq t < q-1$. The quantities $\| u \|$ and $\| Au \|$ remain bounded as $\epsilon \to 0$, the latter since

$$Av \propto \sum b_s P(\xi, z_s)(\xi x + z_s y + \epsilon i)^{\frac{1}{2}-m} \equiv 0.$$

At least one of the numbers b_s, say b_1, does not vanish. Let $Q(\xi, \eta)$ be a homogeneous polynomial having all the (ξ, z_s) but (ξ, z_1) as roots and let N be its corresponding operator. Then

$$Nv \propto b_1 Q(\xi, z_1)(\xi x + z_1 y + \epsilon i)^{\frac{1}{2}-m} \not\equiv 0.$$

Hence $\| Nu \| \to \infty$ as $\epsilon \to 0$ and (IV) cannot hold.

If not all the roots z_s are different, we set $v = \sum Y_s(\xi x + z_s y + \epsilon i)^{\frac{1}{2}}$, where Y_s is a polynomial in y of degree one less than the multiplicity of z_s. One then has enough arbitrary constants to insure that $u \in C^{*r}(G^*)$. The reader is spared the details.

Proof of Corollary 2. Following the proof of Theorem 4, we take $R = \bar{\gamma}\bar{S}$. Then $W = \int |P|^{-2} |S|^2 \operatorname{Re} \gamma P \, d\eta$, and hence $W > 0$ unless $S \equiv 0$. This can happen only if each $g_s = 0$ which in turn implies that each $w_s = 0$ (Lemma 4). The result again follows from Lemma 5.

It now remain only to prove Theorem 2 from Theorem 4. This will be done in the next section.

6. Proof of Theorem 2. We shall show that Theorems 1 and 2 hold for domains G of class C^m. The proof is essentially contained in Lemmas 8-12 which follow. In them we employ inequality (6) due to Ehrling [3] and Nirenberg [6], to which the reader is referred for proof.

Definition. A domain G is said to be in class C^s if every boundary point x of G has a neighborhood $\mathcal{A}$ such that the closure of $\mathcal{A} \cap G$ can be mapped in a one-to-one way onto the closure of a domain G^* of Section 3 with $\mathcal{A} \cap B$ (the boundary of G) mapped into $x_n = 0$ and such that the map and its inverse are s times continuously differentiable.

LEMMA. *If $G \in C^2$ and $s < t$, then for every $\epsilon > 0$ sufficiently small there exists a constant K such that*

$$\| u \|_s^2 \leqq \epsilon \| u \|_t^2 + K \| u \|^2 \quad \textit{for all } u \in C^0(G). \tag{6}$$

Let x be a point in n-space, and assume that A is a linear m-th order partial differential operator with coefficients continuous at x. Denote its characteristic polynomial by P, and let A' be the homogeneous m-th order

operator with constant coefficients which equals the principal part of A at x. We now have

LEMMA 8. *There is a neighborhood $\mathcal{A}$ of x such that*

(III) $$\| u \|_m^2 \leqq K(\| Au \|^2 + \| u \|^2) \text{ for all } u \in U(\mathcal{A})^6$$

if and only if there is a neighborhood $\mathcal{B}$ of x such that

(III′) $$\| u \|_m^2 \leqq K(\| A'u \|^2 + \| u \|^2) \text{ for all } u \in U(\mathcal{B}).$$

Proof. Suppose (III) holds. Since $Au = A'u + (A - A')u$, we have $\| Au \|^2 \leqq 2 \| A'u \|^2 + 2 \| (A - A')u \|^2$. In a small enough neighborhood of x, $\| (A - A')u \|^2 \leqq \epsilon \| u \|_m^2 + K \| u \|_{m-1}^2$, and hence, by (6), for any $\epsilon' > 0$, there is a K such that

$$\| Au \|^2 \leqq 2 \| A'u \|^2 + \epsilon' \| u \|_m^2 + K \| u \|^2$$

in a sufficiently small neighborhood of x. Combining this with (III) shows that we can find a neighborhood $\mathcal{B}$ so small that

$$\| u \|_m^2 \leqq K(\| A'u \|^2 + \| u \|^2) + \tfrac{1}{2} \| u \|_m^2$$

for all $u \in U(\mathcal{B})$. Thus (III′) holds. The converse is proved in the same way.

In light of Lemma 7 we have

LEMMA 9. *There exists a neighborhood $\mathcal{A}$ of x in which* (III) *holds if and only if A is elliptic at x.*

We now suppose that x is on the "flat part" of a domain G^* of Section 3 (i.e. on B^*_1).

LEMMA 10. *There is a neighborhood $\mathcal{A}$ of x such that*

(V) $$\| u \|_m^2 \leqq K(\| Au \|^2 + \| u \|^2) \text{ for all } u \in C^{*r}(\mathcal{A} \cap G^*)$$

if and only if there is a neighborhood $\mathcal{B}$ of x such that

$$\| u \|_m^2 \leqq K(\| A'u \|^2 + \| u \|^2) \text{ for all } u \in C^{*r}(\mathcal{B} \cap G^*).$$

Proof. Same as that for Lemma 8.

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[6] See the beginning of Section 5 for the definition of $U(G)$.

If q is defined for A' as in Section 3, we see from Theorem 4 and its necessity proof that

LEMMA 11. *There exists a neighborhood $\mathcal{A}$ of x in which* (V) *holds if and only if A is elliptic at x and $q \leqq r$.*

We now return to a general domain $G \in C^m$ and to an m-th order operator A with coefficients continuous in $\bar{G}$.

LEMMA 12 (Aronszajn). *A necessary and sufficient condition that* (I) *hold is that each point x of $\bar{G}$ has a neighborhood $\mathcal{A}$ such that*

$$\| u \|_m^2 \leqq K(\| Au \|^2 + \| u \|^2) \text{ for all } u \in [C^r(G) \cap U(\mathcal{A})]. \tag{7}$$

Proof. The necessity is obvious. To show sufficiency, suppose that the condition is satisfied. Thus for each point x of $\bar{G}$, there is a neighborhood $\mathcal{A}$ in which (7) holds. These neighborhoods cover $\bar{G}$, and, by the Heine-Borel theorem, there is a finite subset $(\mathcal{A}_k)$ which does likewise. Let $1 \equiv \sum \zeta_k$ be a partition of unity subordinate to this covering. We may assume that the ζ_k are infinitely differentiable. Thus $u\zeta_k \in [C^r(G) \cap U(\mathcal{A}_k)]$, and hence, by (7), $\| \zeta_k u \|_m^2 \leqq K(\| A\zeta_k u \|^2 + \| u \|^2)$ for all $u \in C^r(G)$. Summing over k, we obtain

$$\sum \| \zeta_k u \|_m^2 \leqq K(\sum \| A\zeta_k u^2 \| + \| u \|^2). \tag{8}$$

Now

$$\| u \|_m^2 = \| \sum \zeta_k u \|_m^2 \leqq K \sum \| \zeta_k u \|_m^2, \tag{9}$$

and since $A\zeta_k u = \zeta_k Au +$ terms of derivatives of u of order $< m$, we have, by (6),

$$\| A\zeta_k u \|^2 \leqq 2 \| \zeta_k Au \|^2 + K \| u \|_{m-1}^2 \leqq 2 \| Au \|^2 + \epsilon \| u \|_m^2 + K \| u \|^2, \tag{10}$$

Inequalities (8), (9) and (10) imply (I). Q.e.d.

We are now ready for the

Proof of Theorem 2. From Lemma 9, we see that ellipticity is essential in the interior of G. Now let x be a point of B, the boundary of G. By hypothesis, there is a neighborhood $\mathcal{A}$ of x such that the closure of $\mathcal{A} \cap G$ can be mapped in a certain way onto the closure of a domain G^* of Section 3. We can make this mapping in two steps. First rotate the coordinate axes until x_n is perpendicular to B at x, and then make the required transformation by changing only x_n but not the other x_k. (This can always be done).

What happens to A? It is transformed into a similar operator A_1 *and the quantity* q *defined for* A *in Section 2 equals the quantity* q *for* A_1 *in Lemma* 11. This in turn is the same for the operator $A_2 = J^{\frac{1}{2}}A_1$, where J is the Jacobian of the transformation. (These facts follow from the ellipticity of A.) We now merely note that (7) holds in $\mathcal{A}$ if and only if

$$\| u \|_m^2 \leqq K(\| A_2 u \|^2 + \| u \|^2) \quad \text{for all} \quad u \in C^{*r}(G^*).$$

The result now follows from Lemmas 11 and 12.

Corollary 1 follows from Corollary 2 in the same way.

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A NOTE ON CONSERVATIVE TRANSFORMATIONS AND THE RECURRENCE THEOREM.*[1]

By Louis Sucheston.

Let X be a space and $\mathcal{B}$ a σ-field of "measurable" subsets of X, that is a class of sets closed under countable set-operations, with $X \in \mathcal{B}$. Let $\mathcal{S}$ be a class of sets of $\mathcal{B}$; we may call elements of $\mathcal{S}$ "sets of measure 0." Assume that a multi-valued transformation T is defined on X, i.e., a function which lets correspond to each point $x \in X$ a non-empty set $Tx \subset X$. The empty set is denoted Θ. $T^{-1}A$ is the set of points x such that $Tx \cap A \neq \Theta$. The transformation T will be always assumed measurable, that is such that $T^{-1}\mathcal{B} \subset \mathcal{B}$. We shall say that T is *conservative* if $A \in \mathcal{S}$ whenever a set $A \in \mathcal{B}$ is disjoint with all sets $T^{-i}A$, $i = 1, 2, \cdots$. It has been proved by Halmos ([1], pp. 736-738, see also [2], p. 13), that any power T^n of a conservative transformation T is again conservative. Halmos assumes that T is single-valued, so that Tx is a single point for each x, that T is one-to-one and that T^{-1} is measurable. The proof, based on an argument of E. Hopf ([3], p. 391), is rather difficult. We propose here a simple proof of this result, which does not make use of the above assumptions. A version of the recurrence theorem extended in the same sense is then derived (compare [1], p. 747), with a corollary about a subsequence of the sequence $T^{-n}A$.

Theorem A. *If $\mathcal{S}$ is closed under finite set-unions and if a transformation T is conservative, then T^n for $n = 1, 2, \cdots$ is conservative.*

Proof. Let $A \in \mathcal{B}$ and

$$A \cap T^{-in}A = \Theta \qquad i = 1, 2, \cdots. \tag{1}$$

We have to show that $A \in \mathcal{S}$. We arrange all sets $T^{-k}A$, $k = 0, 1, \cdots$ in the following scheme:

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$$\begin{array}{lll} A, & T^{-1}A, \cdots, & T^{-(n-1)}A \\ T^{-n}A, & T^{-(n+1)}A, \cdots, & T^{-(2n-1)}A \\ \cdots & \cdots & \cdots \\ T^{-in}A, & T^{-(in+1)}A, \cdots, & T^{-(in+n-1)}A \\ \cdots & \cdots & \cdots \end{array}$$

Let D_j, $j=1,2,\cdots,n$, be the set of points $x \in A$ which belong to exactly j columns of the scheme (2). We shall prove that

$$D_j \cap T^{-v}D_j = 0 \qquad v = 1,2,\cdots \tag{3}$$

and hence $A = \bigcup_{j=1}^{n} D_j \in \mathfrak{B}$. If (3) is not true for some $v \geqq 1$, there exists a point $x \in D_j \cap T^{-v}D_j$. There is a point y such that $y \in T^v x$, $y \in D_j$ and this last relation implies that

$$y \in T^{-n_1}A, \quad y \in T^{-n_2}A, \cdots, \quad y \in T^{-n_j}A \tag{4}$$

where $n_1, \cdots, n_j$ is some set of non-negative integers, different modulo n. Since $x \in T^{-v}y$, we have

$$x \in T^{-(n_1+v)}A, \quad x \in T^{-(n_2+v)}A, \cdots, \quad x \in T^{-(n_j+v)}A \tag{5}$$

where $n_1+v, \cdots, n_j+v$ are *positive* integers, different modulo n. However, we know that $x \in D_j \subset A = T^0 A$ and by (1) that $x \notin T^{-n_i}A$, $i=1,2,\cdots$. Therefore none of the numbers n_s+v, $s=1,2,\cdots,j$ is congruent to 0 modulo n. It follows that x belongs to the sets $T^0A, T^{-(n_1+v)}A, \cdots, T^{-(n_j+v)}A$ which are in $j+1$ different columns. This contradicts $x \in D_j$ and proves the theorem.

We deduce now the recurrence theorem. A point $x \in A$ is called *recurrent* under T (with respect to A) if there exists an infinite set of positive integers n_i such that $T^{n_i}x \cap A \neq 0$ for $i=1,2,\cdots$.

THEOREM B. (*Recurrence Theorem*). *Assume that $\mathfrak{B}$ is closed under countable set-unions, let A be a set of $\mathfrak{B}$ and E the set of points of A not recurrent under a conservative transformation T. Then $E \in \mathfrak{B}$.*

Proof. Put $E_n = \bigcap_{j=0}^{\infty} (A - T^{-(n+j)}A)$. Then for any $v \geqq n > 0$,

$$E_n \subset A - T^{-v}A \text{ and } T^{-v}E_n \subset T^{-v}A,$$

hence $E_n \cap T^{-iv}E_n = 0$ for $i=1,2,\cdots$ and by Theorem A, $E_n \in \mathfrak{B}$. Therefore $E = \bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}$.

It follows also that "almost all" points of A are recurrent under *each* positive power T^n of T. It should be noted that there is a simple proof of the recurrence theorem (but not of the last remark), which does not depend on Theorem A (in [5], where some of Oxtoby's assumptions are not necessary).

We derive still a corollary under the assumption that $\mathcal{S}$ is an *α-ideal* in $\mathcal{B}$, i.e., $\mathcal{S}$ is closed under finite set-unions and $E \cap A \in \mathcal{S}$ whenever $E \in \mathcal{S}$ and $A \in \mathcal{B}$. Let $A \in \mathcal{B}$, it follows from the proof of Theorem B that the set $E_1 = A - \bigcup_{i=1}^{\infty} (A \cap T^{-i}A)$ belongs to $\mathcal{S}$. Hence, if $A \in \mathcal{B} - \mathcal{S}$, there is an integer $r > 0$ such that $A \cap T^{-r}A \in \mathcal{B} - \mathcal{S}$. Put $B = A \cap T^{-r}A$ and consider B and T^{r+1} instead of A and T. There is an integer $s > r$ such that $B \cap T^{-s}B \in \mathcal{B} - \mathcal{S}$, hence $C = A \cap T^{-r}A \cap T^{-s}A \in \mathcal{B} - \mathcal{S}$, for $T^{-s}B \subset T^{-s}A$. Consider now C and T^{s+1}, etc. The argument can be carried on indefinitely. If $A_1, \cdots, A_k \in \mathcal{B}$ and $\bigcap_{i=1}^{k} A_i \in \mathcal{B} - \mathcal{S}$, the collection of sets $A_1, \cdots, A_k$ will be called *compatible*. (The word has its intuitive sense if a probability measure is defined on $\mathcal{B}$ vanishing on $\mathcal{S}$. See then [4] and [6] for additional information.) We have proved the following:

COROLLARY. *Assume that $\mathcal{S}$ is an α-ideal in $\mathcal{B}$, let A be a set, T a conservative transformation and $A_n = T^{-n}A$ for $n = 1, 2, \cdots$. Then for any set $A_k \in \mathcal{B} - \mathcal{S}$ of the sequence $\{A_n\}$, there exists an infinite subsequence $\{A_{n_i}\}$ beginning with A_k and such that any finite collection of sets A_{n_i} is compatible.*

We add the following remarks: Our definition of a conservative transformation T is different from Halmos' definition ([2], p. 12) which requires that $A \in \mathcal{S}$ whenever $A \in \mathcal{B}$ and the sets $T^{-i}A$, $i = 0, 1, \cdots$ are pairwise disjoint. Halmos considers also incompressible transformations: T is incompressible if $T^{-1}A - A \in \mathcal{S}$ whenever $A \in \mathcal{B}$ and $A \subset T^{-1}A$ ([2], p. 13). We assume (to simplify) that $\mathcal{S}$ is an α-ideal in $\mathcal{B}$. The relation between the different considered notions is the following: If a transformation is incompressible, it is conservative in our sense; if a transformation is conservative in our sense, it is conservative in the sense of Halmos. It can be shown by simple examples (and such that a probability measure can be defined on $\mathcal{B}$, vanishing on $\mathcal{S}$), that the inverse implications do not hold, and also that with Halmos' definition the recurrence theorem does not hold for multi-valued transformations. This is of course the reason for which this definition has been here modified. For single-valued transformations (only this case is considered by Halmos and Oxtoby) all three notions coincide.

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P. 20, lines 12 and 23: Instead of "Since $d(d(f\circ\rho_{l-1}{}^{l}))=0$,

$$(22)\qquad D_k\circ D_h = D_h\circ D_k, \qquad (h,k=1,\cdots,p).",$$

read "Since $d(d(f\circ\rho_{l-1}{}^{l}))=0$, for any $h,k=1,\cdots,p$,

$$(22)\qquad D_k\circ D_h f - D_h\circ D_k f \equiv 0 \pmod{\Theta_{\phi;ij;}};\ \phi\in\Sigma^{[2]},\quad i,j=1,\cdots,p)";$$

this should be inserted in the next line of the formula (25).

P. 20, line 6 from the bottom: The formula (26) should be replaced by

$$"(26)\qquad \Theta_{\phi;ij;k_1\cdots k_{l-1}} = \sum (l-1)!^{-1} D_{h_{l-1}}\circ\cdots\circ D_{h_1}(\Theta_{\phi;ij;}),$$

where the summation is with respect to all $(h_1,\cdots,h_{l-1})$ which is a permutation of $(k_1,\cdots,k_{l-1})$."

P. 21, line 3: Instead of "$B_{\phi;ij;\mathfrak{A}_{l-1}}$ are defined inductively by the formula:", read "By (22), it follows that"

P. 21, line 4: Instead of "$B_{\phi;ij;k_1\cdots k_{l-1}}=\ldots$", read

$$"B_{\phi;ij;k_1\cdots k_{l-1}}\equiv\ldots".$$

P. 21, line 6: Insert

$$"(\mathrm{mod}\,\Theta^{\psi}{}_{;fg;h_1\cdots h_{l-3}};\ \psi\in\Sigma^{[2]},f,g,\cdots,h_1,\cdots,h_{l-3}=1,\cdots,p)"$$

at the beginning of the line.

P. 21, line 13: Instead of "$(\mathrm{mod}\,\pi_l)$", read "$(\mathrm{mod}(\Pi_l),\mathfrak{R}_{l-2}{}^{[0]}\Sigma)$".

P. 21, line 14: Insert "$(\mathrm{mod}\,(\Pi_l),\mathfrak{R}_{l-2}{}^{[0]}\Sigma)$".

P. 22, lines 15 and 17: Twice instead of

$$"\mathrm{mod}\,(v^{\lambda}{}_{;k},v^{\lambda}{}_{j_1;k},\cdots,v^{\lambda}{}_{j_1\cdots j_{l-1};k})",$$

read "$(\mathrm{mod}\,v^{\lambda}{}_{;k},v^{\lambda}{}_{j_1;k},\cdots,v^{\lambda}{}_{j_1\cdots j_{l-1};k},v_k{}^{\alpha},\mathfrak{R}_{l-1}{}^{[0]}\Sigma)$".